The effective Vlasov-Poisson system with strong external magnetic field

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Main goals

1. asymptotic analysis for the Vlasov-Poisson equations when disparate scales occur
2. applications to gyro-kinetic theory
3. transport theory
4. strongly anisotropic parabolic problems (F. Filbet, P. Degond, C. Negulescu ...)

Transport equations with disparate advection fields : an example

$$\partial_t u + a(t,y)\cdot\nabla_y u + \omega\,{}^\perp y\cdot\nabla_y u = 0,\ \ (t,y)\in\mathbb{R}_+\times\mathbb{R}^2$$

$$u(0,y) = u^{\mathrm{in}}(y),\ \ y\in\mathbb{R}^2$$

$a\cdot\nabla_y$ advection with $|a|\sim V$, $\omega\,{}^\perp y\cdot\nabla_y$ rotation of frequency ω

Hypothesis

$$\varepsilon = \frac{V}{L\omega} << 1 \text{ that is } \frac{1}{\omega} << \frac{L}{V},\ \ \frac{|\nabla_y u^{\mathrm{in}}|}{|u^{\mathrm{in}}|}\sim\frac{1}{L}$$

Idea

When $\varepsilon\searrow 0$ we average the dynamics of $a\cdot\nabla_y$ w.r.t. the fast rotation generate by $\omega\,{}^\perp y\cdot\nabla_y$

Filtering the fast rotation

$$\frac{\mathrm{d}Y}{\mathrm{d}t} = \omega\,{}^{\perp}Y(t;y), \quad Y(0;y) = y$$

$$u(t,y) = v(t,z), \quad z = Y(-t;y) = \mathcal{R}(\omega t)y$$

$$\partial_t v + \varphi(\omega t)a(t)\cdot\nabla_z v = 0, \quad (t,z) \in \mathbb{R}_+ \times \mathbb{R}^2, \quad v(0,z) = u^{\mathrm{in}}(z), \quad z \in \mathbb{R}^2$$

$$(\varphi(s)c)(z) = \mathcal{R}(s)c(\mathcal{R}(-s)z)$$

Limit model

$$\partial_t v + \langle a(t)\rangle \cdot \nabla_z v = 0, \quad v(0,z) = u^{\mathrm{in}}(z)$$

$$\langle c\rangle = \lim_{S\to+\infty} \frac{1}{S}\int_0^S \varphi(s)c\,\mathrm{d}s = \frac{1}{2\pi}\int_0^{2\pi} \varphi(s)c\,\mathrm{d}s$$

Multi-scale analysis for linear first order PDE

$$\begin{cases} \partial_t u^\varepsilon + a \cdot \nabla_y u^\varepsilon + \frac{1}{\varepsilon} b \cdot \nabla_y u^\varepsilon = 0, & (t,y) \in \mathbb{R}_+ \times \mathbb{R}^m \\ u^\varepsilon(0,y) = u^{\text{in}}(y), & y \in \mathbb{R}^m. \end{cases}$$

Hypotheses

$$a \in L^1_{\text{loc}}(\mathbb{R}_+; W^{1,\infty}_{\text{loc}}(\mathbb{R}^m)), \quad b \in W^{1,\infty}_{\text{loc}}(\mathbb{R}^m) \implies \text{smooth flows}$$

$$\text{div}_y a = 0, \quad \text{div}_y b = 0 \implies \text{measure preserving flows}$$

$$|a(t,y)| + |b(y)| \leq C(1 + |y|) \implies \text{global flows}$$

Vlasov equation with strong magnetic field

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon}(v_1\partial_{x_1} + v_2\partial_{x_2}) f^\varepsilon + v_3\partial_{x_3} f^\varepsilon + \frac{q}{m} E \cdot \nabla_v f^\varepsilon + \frac{qB}{m\varepsilon}(v_2\partial_{v_1} - v_1\partial_{v_2}) f^\varepsilon = 0$$

$$m = 6, \quad y = (x, v), \quad a \cdot \nabla_{x,v} = v_3\partial_{x_3} + \frac{q}{m} E(x) \cdot \nabla_v$$

$$\frac{1}{\varepsilon} b \cdot \nabla_{x,v} = \frac{1}{\varepsilon}(v_1\partial_{x_1} + v_2\partial_{x_2}) + \frac{\omega_c}{\varepsilon}(v_2\partial_{v_1} - v_1\partial_{v_2})$$

Question: behavior when $\varepsilon \searrow 0$?

Main idea: filtering out the fast oscillations

$$\frac{\mathrm{d}Y}{\mathrm{d}s} = b(Y(s;y)), \quad Y(0;y) = y, \quad (s,y) \in \mathbb{R} \times \mathbb{R}^m$$

New coordinates

$$z = Y(-t/\varepsilon, y) \text{ or equivalently } y = Y(t/\varepsilon, z)$$

Search for a profile

$$u^\varepsilon(t,y) = v^\varepsilon(t, \underbrace{Y(-t/\varepsilon; y)}_{z})$$

Why this change of coordinates ?

$$\begin{cases} \partial_t v^\varepsilon(t,z) + \underbrace{\partial_y Y(-t/\varepsilon; Y(t/\varepsilon;z))a(t, Y(t/\varepsilon;z))}_{\varphi(t/\varepsilon)a(t)} \cdot \nabla_z v^\varepsilon(t,z) = 0, \\ v^\varepsilon(0,z) = u^{\mathrm{in}}(z), \end{cases}$$

Stability for $(v^\varepsilon)_{\varepsilon>0}$

$$\varphi(s)a = \partial_y Y(-s; Y(s;\cdot))a(Y(s;\cdot))$$

Behavior when $\varepsilon \searrow 0$

$$\partial_y Y(-t/\varepsilon; Y(t/\varepsilon;z))a(t, Y(t/\varepsilon;z)) = \varphi(t/\varepsilon)a(t)$$

If involution between $a(t)$ and b

$$[b, a(t)] = 0 \implies \varphi(s)a(t) = a(t), s \in \mathbb{R}$$

$$\begin{cases} \partial_t v^\varepsilon(t,z) + a(t,z) \cdot \nabla_z v^\varepsilon(t,z) = 0, & (t,z) \in \mathbb{R}_+ \times \mathbb{R}^m \\ v^\varepsilon(0,z) = u^{\mathrm{in}}(z), & z \in \mathbb{R}^m \end{cases}$$

$$v^\varepsilon(t,z) = u^{\mathrm{in}}(Z(-t;z)) = v(t,z), \quad \frac{\mathrm{d}Z}{\mathrm{d}t} = a(t, Z(t;z))$$

$$u^\varepsilon(t,y) = v(t, Y(-t/\varepsilon; y)) = u^{\mathrm{in}}(Z(-t; Y(-t/\varepsilon; y)))$$

Splitting : advection along a and advection along $\frac{1}{\varepsilon} b$.

Two scale approach : t and $s = t/\varepsilon$

Use ergodicity

$$\lim_{T\to+\infty} \frac{1}{T}\int_0^T \partial_y Y(-s;Y(s;\cdot))a(t,Y(s;\cdot))\,\mathrm{d}s = \lim_{T\to+\infty}\frac{1}{T}\int_0^T \varphi(s)a(t)\,\mathrm{d}s$$

Key point

Emphasize a C^0-group of unitary transformations and use :

von Neumann's Ergodic Mean Theorem

Let $(G(s))_{s\in\mathbb{R}}$ be a C^0-group of unitary operators on a Hilbert space $(H,(\cdot,\cdot))$ and A be the infinitesimal generator of G. Then, for any $x\in H$, we have

$$\lim_{T\to+\infty}\frac{1}{T}\int_r^{r+T} G(s)x\,\mathrm{d}s = \mathrm{Proj}_{\ker A}x, \text{ strongly in } H$$

uniformly with respect to $r\in\mathbb{R}$.

Average vector field

$$X_Q = \{c(y) : \mathbb{R}^m \to \mathbb{R}^m \text{ measurable} : \int_{\mathbb{R}^m} Q(y) : c(y) \otimes c(y)\,\mathrm{d}y < +\infty\}$$

$$Q = P^{-1}, \;\; P = {}^tP > 0, [b, P] := (b \cdot \nabla_y)P - \partial_y b\, P - P\, {}^t\partial_y b = 0$$

$$(c, d)_Q = \int_{\mathbb{R}^m} Q(y) : c(y) \otimes d(y)\,\mathrm{d}y, \;\; c, d \in X_Q.$$

Proposition $(\varphi(s))_{s\in\mathbb{R}}$ is a C^0-group of unitary operators on X_Q.

Theorem

We denote by $\mathcal{L}$ the infinitesimal generator of the group $(\varphi(s))_{s\in\mathbb{R}}$. Then for any vector field $a \in X_Q$, we have the strong convergence in X_Q

$$\langle a \rangle := \lim_{T\to+\infty} \frac{1}{T} \int_r^{r+T} \partial_y Y(-s; Y(s;\cdot))a(Y(s;\cdot))\,\mathrm{d}s = \mathrm{Proj}_{\ker \mathcal{L}} a$$

uniformly with respect to $r \in \mathbb{R}$.

Theorem (Long time behavior)

For any vector field $a \in X_Q$, we consider the problem

$$\begin{cases} \partial_t c - \mathcal{L}^2 c = 0, & t \in \mathbb{R}_+ \\ c(0,\cdot) = a(\cdot) \end{cases}$$

with $\mathcal{L}c = [b,c]$. Then the solution $c(t)$ converges weakly in X_Q, as $t \to +\infty$, toward the orthogonal projection on $\ker \mathcal{L}$

$$\lim_{t \to +\infty} c(t) = \mathrm{Proj}_{\ker \mathcal{L}} a, \text{ weakly in } X_Q.$$

Moreover, if the range of $\mathcal{L}$ is closed, then the previous convergence holds strongly in X_Q and has exponential rate.

Theorem (Convergence)

The family $(v^\varepsilon)_{\varepsilon>0}$ converges strongly in $L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^2(\mathbb{R}^m))$ to a weak solution $v \in L^\infty(\mathbb{R}_+; L^2(\mathbb{R}^m))$ of the transport problem

$$\begin{cases} \partial_t v + \langle a(t,\cdot)\rangle \cdot \nabla_z v = 0, & (t,z) \in \mathbb{R}_+ \times \mathbb{R}^m \\ v(0,z) = u^{\mathrm{in}}(z), & z \in \mathbb{R}^m. \end{cases}$$

Moreover, if v is smooth enough, we have $v^\varepsilon = v + \mathcal{O}(\varepsilon)$ in $L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^2(\mathbb{R}^m))$, as $\varepsilon \searrow 0$.

Formal proof

$$\partial_t v^\varepsilon + \varphi(t/\varepsilon) a(t) \cdot \nabla_z v^\varepsilon = 0$$

$$v^\varepsilon(t,z) = v(t, s = t/\varepsilon, z) + \varepsilon v^1(t, s = t/\varepsilon, z) + \ldots$$

$$\partial_s v = 0, \;\; \partial_t v + \varphi(s) a(t) \cdot \nabla_z v + \partial_s v^1 = 0$$

$$\partial_t v + \left(\lim_{T \to +\infty} \frac{1}{T} \int_0^T \varphi(s) a(t)\, \mathrm{d}s \right) \cdot \nabla_z v = 0.$$

Rigorous proof

Lemma

Let $c \in L^\infty(\mathbb{R}_+; X_Q), d \in L^1(\mathbb{R}_+; X_P)$ such that

$$\exists\, \overline{c} := \lim_{T \to +\infty} \frac{1}{T} \int_r^{r+T} c(s)\, \mathrm{d}s \text{ strongly in } X_Q, \text{ uniformly w.r.t. } r \in \mathbb{R}.$$

Then we have

$$\lim_{\varepsilon \searrow 0} \int_{\mathbb{R}_+} \langle d(t), c(t/\varepsilon) \rangle_{P,Q} \; \mathrm{d}t = \int_{\mathbb{R}_+} \langle d(t), \overline{c} \rangle_{P,Q} \; \mathrm{d}t$$

Application

$$\partial_t v^\varepsilon + \varphi(t/\varepsilon) a(t) \cdot \nabla_z v^\varepsilon = 0$$

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}^m} v^\varepsilon(t,z) \varphi(t/\varepsilon) a(t) \cdot \nabla_z \xi(t,z)\, \mathrm{d}z\mathrm{d}t, \;\; \xi \in C_c^1(\mathbb{R}_+ \times \mathbb{R}^m)$$

$$c(s) = \varphi(s) a, \;\; d(t) = v^\varepsilon(t,\cdot) \nabla_z \xi(t,\cdot), \;\; \lim_{T \to +\infty} \frac{1}{T} \int_r^{r+T} c(s)\, \mathrm{d}s = \langle a \rangle .$$

Convergence rate

$$a(t) = \langle a(t) \rangle + [b, c(t)], \quad t \in \mathbb{R}_+$$

Corrector

$$u^1(t,s,y) = (c(t) \cdot \nabla_z v(t))(Y(-s;y)) - c(t,y) \cdot \nabla_y \{v(t, Y(-s;y))\}$$

$$\widetilde{u}^\varepsilon(t,y) = v(t, Y(-t/\varepsilon; y))$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\{u^\varepsilon - \widetilde{u}^\varepsilon - \varepsilon u^1(t, t/\varepsilon, y)\} + a(t,y) \cdot \nabla_y \{u^\varepsilon - \widetilde{u}^\varepsilon - \varepsilon u^1(t, t/\varepsilon, y)\}$$
$$+ \frac{b}{\varepsilon} \cdot \nabla_y \{u^\varepsilon - \widetilde{u}^\varepsilon - \varepsilon u^1(t, t/\varepsilon, y)\} = -\varepsilon \{\partial_t u^1 + a \cdot \nabla_y u^1\}(t, t/\varepsilon, y)$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \| u^\varepsilon(t,\cdot) - \widetilde{u}^\varepsilon(t,\cdot) - \varepsilon u^1(t, t/\varepsilon, \cdot)\|_{L^2} \leq \varepsilon \|\partial_t u^1(t, t/\varepsilon, \cdot) + a(t,\cdot) \cdot \nabla_y u^1\|_{L^2}$$

Non linear multi-scale problems

Vlasov-Poisson equations

$$\partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon + \frac{q}{m}(-\nabla_x \phi^\varepsilon + v \wedge \mathbf{B}^\varepsilon) \cdot \nabla_v f = 0$$

$$-\varepsilon_0 \Delta_x \phi^\varepsilon = \rho^\varepsilon := q \int f^\varepsilon(t, x, v)\,\mathrm{d}v$$

f^ε : particle presence density in the phase space

$f^\varepsilon(t, x, v)\mathrm{d}x\mathrm{d}v$: particle number inside the volume $\mathrm{d}x\mathrm{d}v$

$E^\varepsilon = -\nabla_x \phi^\varepsilon$: self consistent electric field

Main purposes

Homogenization, two scale convergence

Averaging with respect to the fast cyclotronic motion

Effective Vlasov-Poisson equations

Strong convergence results for any initial conditions (not necessarily well prepared)

General (non uniform) magnetic field

Conservation laws, Hamiltonian structure

The finite Larmor radius regime

The reference time T is much larger than the cyclotronic period (strong magnetic field) *i.e.,*

$$T \frac{q|\mathbf{B}^\varepsilon|}{m} \approx \frac{1}{\varepsilon}, \text{ with } 0 < \varepsilon << 1.$$

The kinetic energy is much larger than the potential energy

$$\frac{m|V|^2}{q\phi} \approx \frac{1}{\varepsilon}$$

The Larmor radius is of the same order as the Debye length *i.e.,*

$$\lambda_D^2 = \frac{\varepsilon_0 \phi}{nq} \approx \rho_L^2.$$

Finite Larmor radius regime

strong uniform magnetic field $\mathbf{B}^\varepsilon = (0, 0, B/\varepsilon)$, perpendicular to $x_1 O x_2$.

$$x = (x_1, x_2), \quad v = (v_1, v_2), \quad {}^\perp v = (v_2, -v_1)$$

$\omega_c = \frac{qB}{m}$ is the rescaled cyclotronic frequency

$T_c = \frac{2\pi}{\omega_c}$ is the rescaled cyclotronic period

the real cyclotronic frequency is $\omega_c^\varepsilon = \omega_c / \varepsilon$

the real cyclotronic period is $T_c^\varepsilon = \frac{2\pi}{\omega_c^\varepsilon} = \varepsilon \frac{2\pi}{\omega_c} = \varepsilon T_c$

References

Frénod, Sonnendrücker, Golse, Han-Kwan, N. Besse, ...

Two dimensional setting

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon}(v \cdot \nabla_x f^\varepsilon + \omega_c \, {}^\perp v \cdot \nabla_v f^\varepsilon) - \nabla_x \phi^\varepsilon \cdot \nabla_v f^\varepsilon = 0, (t,x,v) \in \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R}^2$$

$$-\Delta_x \phi^\varepsilon = \rho^\varepsilon(t,x) := \int_{\mathbb{R}^2} f^\varepsilon(t,x,v)\,\mathrm{d}v, \;\; (t,x) \in \mathbb{R}_+ \times \mathbb{R}^2$$

$$f^\varepsilon(0,x,v) = f^{\mathrm{in}}(x,v), \;\; (x,v) \in \mathbb{R}^2 \times \mathbb{R}^2$$

Main goal

What is the behavior of $(f^\varepsilon, \phi^\varepsilon)_{\varepsilon>0}$ when $\varepsilon \searrow 0$?

What is the limit Vlasov-Poisson system ?

The characteristic equations

The trajectories in (x, v) oscillate at the cyclotronic frequency

$$\frac{\mathrm{d}X^\varepsilon}{\mathrm{d}t} = \frac{V^\varepsilon(t)}{\varepsilon}, \quad \frac{\mathrm{d}V^\varepsilon}{\mathrm{d}t} = \frac{\omega_c}{\varepsilon} \ {}^\perp V^\varepsilon(t) - \nabla_x \phi^\varepsilon(t, X^\varepsilon(t))$$

But $\widetilde{X}^\varepsilon(t) = X^\varepsilon(t) + {}^\perp V^\varepsilon(t)/\omega_c$, $\widetilde{V}^\varepsilon(t) = \mathcal{R}\left(\omega_c t/\varepsilon\right) V^\varepsilon(t)$ are left invariant with respect to the cyclotronic dynamics

$$\frac{\mathrm{d}\widetilde{X}^\varepsilon}{\mathrm{d}t} = -\frac{{}^\perp\nabla_x \phi^\varepsilon(t, X^\varepsilon(t))}{\omega_c}, \quad \frac{\mathrm{d}\widetilde{V}^\varepsilon}{\mathrm{d}t} = -\mathcal{R}\left(\omega_c t/\varepsilon\right) \nabla_x \phi^\varepsilon(t, X^\varepsilon(t))$$

Main idea

stability for $(\widetilde{X}^\varepsilon, \widetilde{V}^\varepsilon)_{\varepsilon > 0}$

$$\frac{\mathrm{d}\widetilde{X}}{\mathrm{d}t} = \mathcal{V}(t, \widetilde{X}(t), \widetilde{V}(t)), \quad \frac{\mathrm{d}\widetilde{V}}{\mathrm{d}t} = \mathcal{A}(t, \widetilde{X}(t), \widetilde{V}(t))$$

Change of phase space coordinates

Presence densities in the phase space $(\widetilde{x}, \widetilde{v})$

$$\tilde{f}^\varepsilon(t, \widetilde{x}, \widetilde{v}) = f^\varepsilon(t, x, v), \;\; \widetilde{x} = x + \frac{^\perp v}{\omega_c}, \;\; \widetilde{v} = \mathcal{R}\left(\omega_c t/\varepsilon\right) v$$

$$\partial_t \tilde{f}^\varepsilon - \frac{^\perp \nabla_x \phi^\varepsilon}{\omega_c} \cdot \nabla_{\widetilde{x}} \tilde{f}^\varepsilon - \mathcal{R}\left(\omega_c t/\varepsilon\right) \nabla_x \phi^\varepsilon \cdot \nabla_{\widetilde{v}} \tilde{f}^\varepsilon = 0, \;\; (t, \widetilde{x}, \widetilde{v}) \in \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R}^2$$

$$\tilde{f}^\varepsilon(0, \widetilde{x}, \widetilde{v}) = f^{\mathrm{in}}\left(\widetilde{x} - \frac{^\perp \widetilde{v}}{\omega_c}, \widetilde{v}\right), \;\; (\widetilde{x}, \widetilde{v}) \in \mathbb{R}^2 \times \mathbb{R}^2.$$

We expect that $\lim_{\varepsilon \searrow 0} \tilde{f}^\varepsilon = \tilde{f}$ and

$$\partial_t \tilde{f} + \mathcal{V} \cdot \nabla_{\widetilde{x}} \tilde{f} + \mathcal{A} \cdot \nabla_{\widetilde{v}} \tilde{f} = 0, \;\; (t, \widetilde{x}, \widetilde{v}) \in \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R}^2$$

The effective trajectories

$$e(z) = -\frac{1}{2\pi} \ln |z|, \quad z \in \mathbb{R}^2 \setminus \{0\}$$

The solution of the Poisson equation

$$\phi^\varepsilon(t,x) = \int_{\mathbb{R}^2} e(x-y)\rho^\varepsilon(t,y)\,\mathrm{d}y = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e(x-y) f^\varepsilon(t,y,w)\,\mathrm{d}w\mathrm{d}y$$

$$\frac{\mathrm{d}\widetilde{X}^\varepsilon}{\mathrm{d}t} = -\frac{{}^\perp\nabla_x \phi^\varepsilon(t, X^\varepsilon(t))}{\omega_c}$$

$$= -\frac{1}{\omega_c}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} {}^\perp\nabla e\left(X^\varepsilon(t) - y\right) f^\varepsilon(t,y,w)\,\mathrm{d}w\mathrm{d}y$$

$$= -\frac{1}{\omega_c}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} {}^\perp\nabla e\left(\widetilde{X}^\varepsilon(t) - \widetilde{y} - \frac{1}{\omega_c}\mathcal{R}\left(-\omega_c t/\varepsilon\right){}^\perp(\widetilde{V}^\varepsilon(t) - \widetilde{w})\right) \tilde{f}^\varepsilon(t,\widetilde{y},\widetilde{w})$$

Average over a cyclotronic period

$$\frac{\widetilde{X}^\varepsilon(t+T_c^\varepsilon) - \widetilde{X}^\varepsilon(t)}{T_c^\varepsilon}$$

$$= -\frac{1}{\omega_c T_c^\varepsilon} \int_t^{t+T_c^\varepsilon} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} {}^\perp\nabla e \left(\widetilde{X}^\varepsilon(\tau) - \widetilde{y} - \mathcal{R}\left(-\frac{\omega_c \tau}{\varepsilon}\right) \frac{{}^\perp(\widetilde{V}^\varepsilon(\tau) - \widetilde{w})}{\omega_c} \right)$$

$$\times \tilde{f}^\varepsilon(\tau, \widetilde{y}, \widetilde{w}) \,\mathrm{d}\tilde{w}\mathrm{d}\tilde{y}\mathrm{d}\tau$$

$$= -\frac{1}{\omega_c} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{1}{2\pi} \int_0^{2\pi} {}^\perp\nabla e \left(\widetilde{X}(t) - \widetilde{y} - \mathcal{R}(\theta) \frac{{}^\perp(\widetilde{V}(t) - \widetilde{w})}{\omega_c} \right) \mathrm{d}\theta \tilde{f} \,\mathrm{d}\tilde{w}\mathrm{d}\tilde{y}$$

$$= -\frac{{}^\perp\nabla_\xi}{\omega_c} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \mathcal{E}(\widetilde{X}(t) - \widetilde{y}, \widetilde{V}(t) - \widetilde{w}) \tilde{f}(t, \widetilde{y}, \widetilde{w}) \,\mathrm{d}\tilde{w}\mathrm{d}\tilde{y} + o(1), \quad \varepsilon \searrow 0$$

$$\mathcal{E}(\xi, \eta) = \frac{1}{2\pi} \int_0^{2\pi} e\left(\xi - \omega_c^{-1} \mathcal{R}(\theta)^\perp \eta\right) \,\mathrm{d}\theta, \quad (\xi, \eta) \in (\mathbb{R}^2 \times \mathbb{R}^2) \setminus \{(0,0)\}$$

Effective velocity field

$$\frac{\mathrm{d}\widetilde{X}}{\mathrm{d}t} = \mathcal{V}[\tilde{f}(t)](\widetilde{X}(t), \widetilde{V}(t))$$

$$\mathcal{V}[\tilde{f}(t)](\widetilde{x}, \widetilde{v}) = -\frac{{}^{\perp}\nabla_{\xi}}{\omega_c} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \mathcal{E}(\widetilde{x}-\widetilde{y}, \widetilde{v}-\widetilde{w})\tilde{f}(t, \widetilde{y}, \widetilde{w}) \,\mathrm{d}\tilde{w}\mathrm{d}\tilde{y}, (\widetilde{x}, \widetilde{v}) \in \mathbb{R}^4$$

Effective acceleration field

$$\frac{\mathrm{d}\widetilde{V}}{\mathrm{d}t} = \mathcal{A}[\tilde{f}(t)](\widetilde{X}(t), \widetilde{V}(t))$$

$$\mathcal{A}[\tilde{f}(t)](\widetilde{x}, \widetilde{v}) = \omega_c\, {}^{\perp}\nabla_{\eta} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \mathcal{E}(\widetilde{x}-\widetilde{y}, \widetilde{v}-\widetilde{w})\tilde{f}(t, \widetilde{y}, \widetilde{w}) \,\mathrm{d}\tilde{w}\mathrm{d}\tilde{y}, (\widetilde{x}, \widetilde{v}) \in \mathbb{R}^4$$

The function $\mathcal{E}$

$\mathcal{E}(\xi,\eta)$ is the average of the fundamental solution $e(\cdot)$ over the circle of center ξ and radius $|\eta|/|\omega_c|$

If $|\xi| > |\eta|/|\omega_c|$, the function $z \to e(z)$ is harmonic in the open set $\mathbb{R}^2 \setminus \{0\}$, which contains the disc $\{z \in \mathbb{R}^2 : |z-\xi| \le |\eta|/|\omega_c|\}$ and thus, the mean property applied to the function $e(\cdot)$ and the circle of center ξ and radius $|\eta|/|\omega_c|$ yields

$$\mathcal{E}(\xi,\eta) = \frac{1}{2\pi}\int_0^{2\pi} e\left(\xi - \frac{\mathcal{R}(\theta)}{\omega_c}\,{}^\perp\eta\right)\,\mathrm{d}\theta = e(\xi) = -\frac{1}{2\pi}\ln|\xi|,\ \ |\xi| > \frac{|\eta|}{|\omega_c|}.$$

The function $\mathcal{E}$

The function $\mathcal{E}$ has also the symmetry property $\mathcal{E}(\omega_c^{-1}\eta, \omega_c\xi) = \mathcal{E}(\xi, \eta)$ for any $(\xi, \eta) \in (\mathbb{R}^2 \times \mathbb{R}^2) \setminus \{(0,0)\}$.
If $|\xi| < |\eta|/|\omega_c|$, then

$$\mathcal{E}(\xi, \eta) = \mathcal{E}\left(\frac{\eta}{\omega_c}, \omega_c\xi\right) = e\left(\frac{\eta}{\omega_c}\right)$$

Finally

$$\mathcal{E}(\xi, \eta) = e\left(\frac{\eta}{\omega_c}\right)\mathbf{1}_{\{|\xi| \leq |\eta|/|\omega_c|\}} + e(\xi)\mathbf{1}_{\{|\xi| > |\eta|/|\omega_c|\}}.$$

The velocity and acceleration fields associated to any density $\tilde{f} = \tilde{f}(\widetilde{x}, \widetilde{v})$ write

$$\mathcal{V}[\tilde{f}](\widetilde{x}, \widetilde{v}) = \frac{1}{2\pi\omega_c} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{{}^\perp(\widetilde{x} - \widetilde{y})}{|\widetilde{x} - \widetilde{y}|^2} \tilde{f}(\widetilde{y}, \widetilde{w}) \mathbf{1}_{\{|\widetilde{x}-\widetilde{y}| > \frac{|\widetilde{v}-\widetilde{w}|}{|\omega_c|}\}} \,\mathrm{d}\tilde{w}\mathrm{d}\tilde{y}$$

$$\mathcal{A}[\tilde{f}](\widetilde{x}, \widetilde{v}) = -\frac{\omega_c}{2\pi} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{{}^\perp(\widetilde{v} - \widetilde{w})}{|\widetilde{v} - \widetilde{w}|^2} \tilde{f}(\widetilde{y}, \widetilde{w}) \mathbf{1}_{\{|\widetilde{x}-\widetilde{y}| \leq \frac{|\widetilde{v}-\widetilde{w}|}{|\omega_c|}\}} \,\mathrm{d}\tilde{w}\mathrm{d}\tilde{y}.$$

Non linear limit model

$$\partial_t \tilde{f} + \mathcal{V}[\tilde{f}](\widetilde{x}, \widetilde{v}) \cdot \nabla_{\widetilde{x}} \tilde{f} + \mathcal{A}[\tilde{f}](\widetilde{x}, \widetilde{v}) \cdot \nabla_{\widetilde{v}} \tilde{f} = 0.$$

Conservations

1. Let $\tilde{f} = \tilde{f}(t, \widetilde{x}, \widetilde{v})$ be a solution of the limit problem such that $1, \widetilde{x}, \widetilde{v}, |\widetilde{x}|^2, |\widetilde{v}|^2$ are integrable functions with respect to $\tilde{f}(0, \widetilde{x}, \widetilde{v})\mathrm{d}\widetilde{v}\mathrm{d}\widetilde{x} = f^{\mathrm{in}}(\widetilde{x} - \omega_c^{-1\perp}\widetilde{v}, \widetilde{v})\mathrm{d}\widetilde{v}\mathrm{d}\widetilde{x}$. For any $t \in \mathbb{R}_+$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\{1, \widetilde{x}, \widetilde{v}, |\widetilde{x}|^2, |\widetilde{v}|^2\}\tilde{f}(t, \widetilde{x}, \widetilde{v})\,\mathrm{d}\tilde{v}\mathrm{d}\tilde{x} = 0$$

2. Let $\tilde{f} = \tilde{f}(t, \widetilde{x}, \widetilde{v})$ be a solution of the limit problem such that

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\mathcal{E}(\widetilde{x}-\widetilde{y}, \widetilde{v}-\widetilde{w})\tilde{f}(0, \widetilde{y}, \widetilde{w})\tilde{f}(0, \widetilde{x}, \widetilde{v})\,\mathrm{d}\tilde{w}\mathrm{d}\tilde{y}\,\mathrm{d}\tilde{v}\mathrm{d}\tilde{x} < +\infty.$$

The electric energy is preserved in time

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\tilde{\phi}[\tilde{f}(t)](\widetilde{x}, \widetilde{v})\tilde{f}(t, \widetilde{x}, \widetilde{v})\,\mathrm{d}\tilde{v}\mathrm{d}\tilde{x} = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\tilde{\phi}[\tilde{f}(t)]\partial_t\tilde{f}\,\mathrm{d}\tilde{v}\mathrm{d}\tilde{x} = 0.$$

Lemma

Let $\tilde{f} = \tilde{f}(t, \widetilde{x}, \widetilde{v})$ be a solution of the limit model and $\psi = \psi(\widetilde{x}, \widetilde{v})$ be a smooth integrable function with respect to $\tilde{f}(0, \widetilde{x}, \widetilde{v})\mathrm{d}\widetilde{v}\mathrm{d}\widetilde{x} = f^{\mathrm{in}}(\widetilde{x} - \omega_c^{-1\perp}\widetilde{v}, \widetilde{v})\mathrm{d}\widetilde{v}\mathrm{d}\widetilde{x}$. For any $t \in \mathbb{R}_+$ we have

$$2\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\psi(\widetilde{x},\widetilde{v})\tilde{f}(t,\widetilde{x},\widetilde{v})\,\mathrm{d}\tilde{v}\mathrm{d}\tilde{x} = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\tilde{f}(t,\widetilde{y},\widetilde{w})\tilde{f}(t,\widetilde{x},\widetilde{v})$$
$$\times\left[\frac{1}{\omega_c}\left(\nabla_{\widetilde{y}}\psi(\widetilde{y},\widetilde{w}) - \nabla_{\widetilde{x}}\psi(\widetilde{x},\widetilde{v})\right)\cdot {}^{\perp}\nabla e(\widetilde{x}-\widetilde{y})\,\mathbf{1}_{\{|\widetilde{x}-\widetilde{y}|>|\widetilde{v}-\widetilde{w}|/|\omega_c|\}}\right.$$
$$\left. + \left(\nabla_{\widetilde{v}}\psi(\widetilde{x},\widetilde{v}) - \nabla_{\widetilde{w}}\psi(\widetilde{y},\widetilde{w})\right)\cdot {}^{\perp}\nabla e\left(\frac{\widetilde{v}-\widetilde{w}}{\omega_c}\right)\mathbf{1}_{\{|\widetilde{x}-\widetilde{y}|<|\widetilde{v}-\widetilde{w}|/|\omega_c|\}}\right]$$

Hamiltonian formulation of the Vlasov equation

Scalar and vector potentials

$E = -\nabla_x \phi, \quad \mathbf{B} = \nabla_x \wedge A$

Symplectic structure on $\mathbb{R}^6$

$$\theta(\xi, \eta) = q\mathbf{B} \cdot (\xi_x \wedge \eta_x) + m(\xi_v \cdot \eta_x - \xi_x \cdot \eta_v)$$

$$\theta = \mathrm{d}(qA\,\mathrm{d}x + mv\,\mathrm{d}x)$$

Hamiltonian function

$$H = \frac{m|v|^2}{2} + q\phi$$

Hamiltonian vector field

$$\mathrm{d}H(\cdot) = \theta(\cdot, X), \quad X = v \cdot \nabla_x + \frac{q}{m}(E + v \wedge \mathbf{B}) \cdot \nabla_v$$

Vlasov equation

$$\partial_t f + [\ H[f(t)], f(t)] = 0$$

$$H[f] = \frac{m|v|^2}{2} + \frac{q^2}{\varepsilon_0} \int\int \frac{f(y,w)}{4\pi|x-y|}\,\mathrm{d}w\mathrm{d}y$$

$$[H,f] = \frac{\nabla_v H}{m} \cdot \nabla_x f - \frac{\nabla_x H}{m} \cdot \nabla_v f + (\nabla_v f \wedge \nabla_v H) \cdot \frac{q\mathbf{B}}{m}$$

Total energy conservation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{ \int\int \frac{m|v|^2}{2} f(t,x,v)\,\mathrm{d}v\mathrm{d}x + \frac{q^2}{2\varepsilon_0} \int\int\int\int \frac{f(t,x,v)f(t,y,w)}{4\pi|x-y|} \right\} = 0$$

Uniform magnetic field

E. Sonnendrücker, E, Frénod, M. Lemou, N. Crouseilles, F. Golse, L. Saint-Raymond, ...

$$\partial_t f^\varepsilon + v_3 \partial_{x_3} f^\varepsilon + \frac{q}{m} E(t,x) \cdot \nabla_v f^\varepsilon + \frac{1}{\varepsilon}(v_1 \partial_{x_1} + v_2 \partial_{x_2} + v_2 \partial_{v_1} - v_1 \partial_{v_2}) f^\varepsilon = 0$$

$$a(t,x) \cdot \nabla_{x,v} = v_3 \partial_{x_3} + \frac{q}{m} E(t,x) \cdot \nabla_v$$

$$b \cdot \nabla_{x,v} = v_1 \partial_{x_1} + v_2 \partial_{x_2} + v_2 \partial_{v_1} - v_1 \partial_{v_2}$$

Non uniform magnetic field

$\mathbf{B} = B(x) e(x),\ B(x) > 0,\ |e(x)| = 1$

$$a(t,x) \cdot \nabla_{x,v} = (v \cdot e(x)) e(x) \cdot \nabla_x + \frac{q}{m} E(t,x) \cdot \nabla_v$$

$$b \cdot \nabla_{x,v} = [v - (v \cdot e(x)) e(x)] \cdot \nabla_x + \omega(x)(v \wedge e(x)) \cdot \nabla_v, \ \ \omega(x) = \frac{qB(x)}{m}$$

Parallel perpendicular kinetic energies

$$H = H_a + H_b, \quad H_a = \frac{m(v\cdot e(x))^2}{2} + q\phi, \quad H_b = \frac{m|v\wedge e(x)|^2}{2}$$

Hamiltonian vector fields

$$a\cdot\nabla_{x,v} = (v\cdot e(x))e(x)\cdot\nabla_x + \left[\frac{q}{m}E - (v\cdot e(x))\ {}^t\partial_x e\, v\right]\cdot\nabla_v$$

$$b\cdot\nabla_{x,v} = [v-(v\cdot e(x))e(x)]\cdot\nabla_x + \left[\omega(x) v\wedge e(x) + (v\cdot e(x))\ {}^t\partial_x e\, v\right]\cdot\nabla_v$$

$$\omega(x) = \frac{q\,B(x)}{m}$$

$$\mathrm{div}_{x,v} a = \mathrm{div}_{x,v} b = 0$$

Lemma Let b, c be two hamiltonians vector fields on the symplectic manifold $(\mathbb{R}^m, \theta)$, corresponding to the Hamiltonians H_b, H_c. Then the average vector field (along the flow Y of b)

$$\langle c \rangle = \lim_{T \to +\infty} \frac{1}{T} \int_0^T \partial Y(-t; Y(t;\cdot)) c(Y(t;\cdot)) \, \mathrm{d}t$$

is hamiltonian and corresponds to the average Hamiltonian

$$\langle H_c \rangle = \lim_{T \to +\infty} \frac{1}{T} \int_0^T H_c(Y(t;\cdot)) \, \mathrm{d}t.$$

Proof : use the invariance of the symplectic form along hamiltonian fields.

Advantages

Emphasize the hamiltonian structure of the effective Vlasov-Poisson equations

Computing $\langle a \rangle$ requires 6 averages. But if hamiltonian field, one average is enough!

Average of $a \cdot \nabla_{x,v}$ w.r.t. $b \cdot \nabla_{x,v}$

$$f^\varepsilon(t,x,v) = F^\varepsilon(t, \mathcal{X}(-t;x,v), \mathcal{V}(-t;x,v))$$

$$\frac{\mathrm{d}\mathcal{X}}{\mathrm{d}t} = b_x(\mathcal{X}(t), \mathcal{V}(t)), \quad \frac{\mathrm{d}\mathcal{V}}{\mathrm{d}t} = b_v(\mathcal{X}(t), \mathcal{V}(t))$$

$$\partial_t F^\varepsilon + \varphi(t) a^\varepsilon(t) \cdot \nabla_{X,V} F^\varepsilon = 0$$

$$H_a(t, (\mathcal{X}, \mathcal{V})(t;x,v)) = \frac{m(\mathcal{V} \cdot e)^2}{2} + \frac{q^2}{\varepsilon_0} \int\int \frac{F(t,Y,W)\,\mathrm{d}W\mathrm{d}Y}{4\pi|\mathcal{X}(t;X,V) - \mathcal{X}(t;Y,W)|}$$

$$\langle H_a \rangle (X,V) = m \frac{\left\langle (v \cdot e)^2 \right\rangle}{2} + \frac{q^2}{\varepsilon_0} \int\int \mathcal{E}(X,V,Y,W) F(Y,W)\,\mathrm{d}W\mathrm{d}Y$$

$$\mathcal{E}(X,V,Y,W) = \lim_{T \to +\infty} \frac{1}{T} \int_0^T \frac{\mathrm{d}t}{4\pi|\mathcal{X}(t;X,V) - \mathcal{X}(t;Y,W)|}.$$

Limit model

$$\partial_t F + [\,\mathcal{H}[F(t)], F(t)\,] = 0$$

$$\mathcal{H}[F] = m\frac{\langle (v\cdot e)^2\rangle}{2} + \frac{q^2}{\varepsilon_0}\int\!\!\int \mathcal{E}(X,V,Y,W)F(Y,W)\,\mathrm{d}W\mathrm{d}Y$$

Conservations

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\!\!\int F(t,X,V)\,\mathrm{d}V\mathrm{d}X = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\!\!\int F(t,X,V)\frac{m|V\wedge e(X)|^2}{2}\,\mathrm{d}V\mathrm{d}X = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\!\!\int F(t,X,V)\left\{\frac{m\,\langle (v\cdot e)^2\rangle}{2} + \frac{q^2}{2\varepsilon_0}\int\!\!\int \mathcal{E}F\,\mathrm{d}W\mathrm{d}Y\right\}\mathrm{d}V\mathrm{d}X = 0$$

Exact computation of $\mathcal{E}$, 2D, uniform magnetic field

$$L_2(z) = -\frac{1}{2\pi} \ln |z|, \;\; z \in \mathbb{R}^2, \;\; z \neq 0$$

$$\mathcal{E} = -\frac{1}{2\pi T_c} \int_0^{T_c} \ln \left| X - Y + \frac{{}^\perp(V-W)}{\omega_c} - \frac{\mathcal{R}(-\omega_c t)}{\omega_c} {}^\perp(V-W) \right| \mathrm{d}t$$

$$\mathcal{E} = -\frac{1}{2\pi} \ln \left| X - Y + \frac{{}^\perp(V-W)}{\omega_c} \right|, \; \left| X - Y + \frac{{}^\perp(V-W)}{\omega_c} \right| > \left| \frac{V-W}{\omega_c} \right|$$

$$\mathcal{E} = -\frac{1}{2\pi} \ln \left| \frac{{}^\perp(V-W)}{\omega_c} \right|, \; \left| X - Y + \frac{{}^\perp(V-W)}{\omega_c} \right| < \left| \frac{V-W}{\omega_c} \right|$$

Collisions (2D, uniform magnetic field)

- gyro-kinetic models with collisional effects
- average collision kernels
- collisional equilibria, collisional invariants

Collision kernel

Boltzmann, Fokker-Planck, Landau-Fokker-Planck

Theorem H

$\int Q(f)(v) \ln f(v) \, \mathrm{d}v \leq 0$

$\int Q(f)(v) \ln f(v) \, \mathrm{d}v = 0$ iff $Q(f) = 0$ iff $\ln f \in \mathrm{span}\{1, v, |v|^2/2\}$.

Average collision kernel

$$\langle Q \rangle (F) = \lim_{T \to +\infty} \frac{1}{T} \int_0^T Q(F_{-t})_t \, \mathrm{d}t$$

Notation : $G_t = G(\mathcal{X}(t;\cdot,\cdot), \mathcal{V}(t;\cdot,\cdot))$

Proposition (2D uniform magnetic field)

If Q is local in space, then $\langle Q \rangle$ is local in the parallel space coordinate

$$(F - G)(\cdot,\cdot,X_3,\cdot,\cdot,\cdot) = 0 \implies (\langle Q \rangle (F) - \langle Q \rangle (G))(\cdot,\cdot,X_3,\cdot,\cdot,\cdot) = 0$$

Remark If $\int Q(f) \, \mathrm{d}v = 0$ for any $f = f(v)$, then
$\int \langle Q \rangle (F) \, \mathrm{d}V \mathrm{d}X_1 \mathrm{d}X_2 = 0$ for any $F = F(X_1, X_2, V)$.

Equilibria/invariants of $\langle Q \rangle$ A function $C = C(X, V)$ is a collisional invariant for $\langle Q \rangle$ iff C_{-t} is a collisional invariant for Q, for any $t \in \mathbb{R}$. A presence density $F = F(X, V)$ is an equilibrium for $\langle Q \rangle$ iff F_{-t} is an equilibrium for Q, for any $t \in \mathbb{R}$.

Proof

$$\int \langle Q \rangle (F) \ln F \mathrm{d}V \mathrm{d}X = \frac{1}{T_c} \int_0^{T_c} \int Q(F_{-t}) \ln F_{-t} \, \mathrm{d}V \mathrm{d}X \mathrm{d}t$$

Collisional invariants

$$1, x_1 + \frac{v_2}{\omega_c}, x_2 - \frac{v_1}{\omega_c}, v_1, v_2, v_3, \frac{|v|^2}{2}, \left| (x_1, x_2) + \frac{(v_2, -v_1)}{\omega_c} \right|^2 - \frac{|(v_1, v_2)|^2}{\omega_c^2}.$$

Perspectives

1. limit models with non uniform magnetic fields
2. gyrokinetic collisional models, average collision operators, invariants, equilibria (Ph.D. A. Finot)
3. strongly anisotropic parabolic problems (strong convergence results for initial conditions not necessarily well prepared, Ph. D. T. Blanc)

$$\begin{cases} \partial_t u^\varepsilon - \mathrm{div}_y(D(y)\nabla_y u^\varepsilon) - \frac{1}{\varepsilon}\mathrm{div}_y(b(y)\otimes b(y)\nabla_y u^\varepsilon) = 0, \\ u^\varepsilon(0,y) = u^{\mathrm{in}}(y). \end{cases}$$