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Singular limits of anisotropic weak solutions to compressible magnetohydrodynamics

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ABSTRACT

The aim is to justify rigorously the so-called reduced magnetohydrodynamic model (abbreviated as RMHD), which is widely used in fusion, space and astrophysical plasmas. Motivated by physics, the focus is on plasmas that are simultaneously strongly magnetized and anisotropic. We consider conducting fluids that can be described by viscous and resistive barotropic compressible magnetohydrodynamic equations. The purpose is to study the asymptotic behavior of global weak solutions, which do exist, for strongly anisotropic plasmas such as the large aspect ratio framework. We prove that such anisotropic weak solutions converge to the weak solutions of the RMHD equations. Rigorous justification of this limit is performed both in a periodic domain and in the whole space. It turns out that the resulting system is incompressible only in the perpendicular direction to the external strong magnetic field, whereas it involves compressible features in the parallel direction. In order to pass to the singular limit in the perpendicular direction we exploit, among others, tools elaborated for proving the low Mach number limit of compressible neutral fluid flows such as, here, the introduction of a fast oscillatory unitary group associated to the dynamics of transverse fast magnetosonic waves. In the parallel direction, we bring out compactness arguments and particular cancellations coming from the structure of our equations.

RÉSUMÉ

L'objectif de ce travail est de justifier rigoureusement le modèle de la magnétohydrodynamique réduite (en abrégé RMHD) qui est abondamment utilisé dans les plasmas de fusion, spatiaux et astrophysiques. Motivés par des considérations physiques, nous nous concentrons sur des plasmas qui sont à la fois fortement magnétisés et anisotropes. Plus précisément, nous considérons des fluides conducteurs qui peuvent être décrits par les équations de la magnétohydrodynamique compressible barotrope, visqueuse et résistive. Le but est alors d'étudier le comportement asymptotique des solutions faibles globales, qui existent, pour des plasmas fortement anisotropes tels que ceux à grand rapport d'aspect. Nous prouvons que ces solutions faibles anisotropes convergent vers les solutions faibles des équations de la RMHD. Une

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justification rigoureuse de cette limite est effectuée à la fois sur un domaine périodique et dans l'espace entier. Il s'avère que le système limite obtenu n'est incompressible que dans la direction perpendiculaire au champ magnétique fort externe, tandis qu'il présente des caractéristiques compressibles dans la direction parallèle. Afin de passer à la limite singulière dans la direction perpendiculaire, nous exploitons, entre autres, des outils élaborés pour prouver la limite de faible nombre de Mach des écoulements de fluides neutres compressibles, tels qu'ici l'introduction d'un groupe unitaire, fortement oscillant, associé à la dynamique des ondes magnétosoniques transverses rapides. Dans la direction parallèle, nous mettons en oeuvre des arguments de compacité et des compensations particulières provenant de la structure des équations.

Contents

1. Introduction

Equations of reduced magnetohydrodynamics, hereafter abbreviated as RMHD, are extensively used in fusion, space and astrophysical plasmas. They are highly prized by plasma physicists for the following reasons. First, they allow interesting theoretical and analytical developments; second they are the source of numerically tractable models which are used to bring forth codes that are routinely exploited [3,10,32,36–39]. The RMHD model was introduced in the seventhies [26,48] in the context of fusion plasmas. It was followed by many systematic studies and generalizations including more and more physical effects and refinements [12,13,22,24,29,32,49,51,52]. At present, there is a vast literature about formal derivations and applications of RMHD models. The two references [4,41] are good introductions to the subject, with many references. RMHD equations are still a very active research field, including recent progress in extended magnetohydrodynamics [1]. At the same time, the work [2] has highlighted the importance and the inherent difficulties of working under anisotropic conditions.

1.1. The penalized system

Let Ω be a three-dimensional domain, which is either the periodic box $\mathbb{T}^3 := (\mathbb{R}/2\pi\mathbb{Z})^3$ or the whole space $\mathbb{R}^3$. The time evolution on Ω of plasmas is basically described by isentropic compressible magnetohydrodynamics. The unknowns are made of the fluid density $\rho \in \mathbb{R}^+$, the fluid velocity $v \in \mathbb{R}^3$ and the magnetic field $B \in \mathbb{R}^3$. Including viscous ($\mu > 0$ and $\lambda > 0$) and resistive ($\eta > 0$) effects, we consider on $\mathbb{R}_+ \times \Omega$ the system of MHD equations

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho\, v) = 0\,, \\ \partial_t(\rho\, v) + \nabla \cdot (\rho\, v \otimes v) + \nabla p + B \times (\nabla \times B) - \mu\, \Delta v - \lambda\, \nabla(\nabla \cdot v) = 0\,, \\ \partial_t B + \nabla \times (B \times v) - \eta\, \Delta B = 0\,, \end{cases} \tag{1}$$

together with the divergence-free condition $\nabla \cdot \mathrm{B} = 0$ and the barotropic state law $\mathrm{p} = \mathfrak{a}\rho^\gamma$, where $\mathfrak{a} > 0$ and $\gamma > 1$. The symbol $\times$ denotes the cross product of vectors of $\mathbb{R}^3$, while the symbol $\otimes$ stands for the tensor product of vectors. Given $(n, m) \in \mathbb{N}_*^2$, we take the convention

$$\big(\nabla \cdot (u \otimes v)\big)_i = \sum_{j=1}^{n} \partial_j (u_i v_j) \,, \quad u = {}^t(u_1, \cdots, u_m) \in \mathbb{R}^m \,, \quad v = {}^t(v_1, \cdots, v_n) \in \mathbb{R}^n \,.$$

Following preceding results about compressible Navier–Stokes equations [15,33], the existence of global (in time) weak solutions to system (1) has been obtained in [23]. Now, most plasmas are magnetized. This means that B is, in a first rough approximation, a given external non-zero magnetic field $\mathrm{B}_{\mathfrak{e}}$. For simplicity, we assume that $\mathrm{B}_{\mathfrak{e}}$ is constant. After rotation, it can always be adjusted in such a way that $\mathrm{B}_{\mathfrak{e}} = \mathbb{B}\, e_\parallel$ with $\mathbb{B} \in \mathbb{R}_+^*$ and $e_\parallel := {}^t(0, 0, 1)$. The direction $e_\parallel$ is called *parallel*. Given a vector field like B (or v), we can decompose B into its parallel component $\mathrm{B}_\parallel = \mathrm{B}_3 := \mathrm{B} \cdot e_\parallel \in \mathbb{R}$ and its perpendicular component $\mathrm{B}_\perp := {}^t(\mathrm{B}_1, \mathrm{B}_2) \in \mathbb{R}^2$ so that $\mathrm{B} = {}^t({}^t\mathrm{B}_\perp, \mathrm{B}_\parallel)$. We work away from vacuum, near a constant density which can always be put in the form $\mathbb{B}^2\, \wp$ for some $\wp \in \mathbb{R}_+^*$. Observe that $(\mathbb{B}^2\, \wp, 0, \mathrm{B}_{\mathfrak{e}})$ is a constant solution to (1). Motivated by physics, particularly by considerations of large aspect ratio and geometrical optics (see Section 3), we incorporate a strong spatial anisotropy. More precisely, we keep $\mathrm{x}_\perp := (\mathrm{x}_1, \mathrm{x}_2) = x_\perp = (x_1, x_2)$ and we replace the vertical direction by $x_\parallel = x_3 := \varepsilon\, \mathrm{x}_3$ with $0 < \varepsilon \ll 1$. The above gradient operator becomes

$$\nabla_\varepsilon := {}^t({}^t\nabla_\perp, 0) + \varepsilon\, \nabla_\parallel \,, \qquad \nabla_\perp := {}^t(\partial_1, \partial_2) \,, \qquad \nabla_\parallel := e_\parallel\, \partial_\parallel \,, \qquad \partial_\parallel \equiv \partial_3 \,. \tag{2}$$

Accordingly, a distinction must be drawn between $\Delta_\perp := \partial_{11}^2 + \partial_{22}^2$ and $\Delta_\parallel := \partial_{33}^2$. We want to study the behavior at large time scales $t := \varepsilon\, \mathrm{t} \sim 1$ of small perturbations, of size ε, of the stationary solution $(\mathbb{B}^2\, \wp, 0, \mathrm{B}_{\mathfrak{e}})$. To this end, we seek solutions in the form

$$(\rho, \mathrm{v}, \mathrm{B})(\mathrm{t}, \mathrm{x}) = \big(\mathbb{B}^2\, \rho^\varepsilon, \, \varepsilon\, v^\varepsilon, \, \mathbb{B}\, (e_\parallel + \varepsilon\, B^\varepsilon)\big)(\varepsilon \mathrm{t}, x_1, \mathrm{x}_2, \varepsilon \mathrm{x}_3) \,.$$

The unknowns are now $(\rho^\varepsilon, v^\varepsilon, B^\varepsilon)(t, x)$, while the pressure is given by $p^\varepsilon = a\, (\rho^\varepsilon)^\gamma$, with $a = \mathbb{B}^{2(\gamma-1)}\mathfrak{a}$. Then, the system (1) can be reformulated according to the following equations

$$\begin{cases} \partial_t \rho^\varepsilon + \nabla_\varepsilon \cdot (\rho^\varepsilon v^\varepsilon) = 0 \,, \\ \partial_t (\rho^\varepsilon v^\varepsilon) + \nabla_\varepsilon \cdot (\rho^\varepsilon v^\varepsilon \otimes v^\varepsilon) + \dfrac{1}{\varepsilon^2}\, \nabla_\varepsilon p^\varepsilon + \dfrac{1}{\varepsilon} (e_\parallel + \varepsilon B^\varepsilon) \times (\nabla_\varepsilon \times B^\varepsilon) \\ \qquad\qquad - \mu_\perp^\varepsilon \Delta_\perp v^\varepsilon - \mu_\parallel^\varepsilon \Delta_\parallel v^\varepsilon - \lambda^\varepsilon \nabla_\varepsilon (\nabla_\varepsilon \cdot v^\varepsilon) = 0 \,, \\ \partial_t B^\varepsilon + \dfrac{1}{\varepsilon} \nabla_\varepsilon \times \big((e_\parallel + \varepsilon B^\varepsilon) \times v^\varepsilon\big) - \eta_\perp^\varepsilon \Delta_\perp B^\varepsilon - \eta_\parallel^\varepsilon \Delta_\parallel B^\varepsilon = 0 \,, \end{cases} \tag{3}$$

together with

$$\nabla_\varepsilon \cdot B^\varepsilon = 0 \,. \tag{4}$$

Without loss of generality, just to simplify the presentation, we can work with $\wp = 1$. The second equation (for the momentum $\rho^\varepsilon v^\varepsilon$) in (3) indicates that ρ^ε should be like $\rho^\varepsilon = \wp + \mathcal{O}(\varepsilon)$. With this in mind, we can introduce the new state variable ϱ^ε as indicated below, and expand p^ε in powers of ε to obtain

$$\rho^\varepsilon = 1 + \varepsilon\, \varrho^\varepsilon \,, \qquad p^\varepsilon = a + \varepsilon\, \mathbb{p}^\varepsilon + \mathcal{O}(\varepsilon^2) \,, \qquad \mathbb{p}^\varepsilon := b\, \varrho^\varepsilon \,, \qquad b := a\, \gamma \,. \tag{5}$$

To see the heuristics which lead to our model, it is instructive to interpret (3) in terms of $(\varrho^\varepsilon, v^\varepsilon, B^\varepsilon)$, and then to extract the singular part. We find

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$$\begin{cases} \partial_t \varrho^\varepsilon + \dfrac{1}{\varepsilon}\, \nabla_\perp \cdot v_\perp^\varepsilon = \mathcal{O}(1)\,, \\ \partial_t v_\perp^\varepsilon + \dfrac{1}{\varepsilon}\, \nabla_\perp (\wp^\epsilon + B_\parallel^\varepsilon) = \mathcal{O}(1)\,, \qquad \partial_t v_\parallel^\varepsilon = \mathcal{O}(1)\,, \\ \partial_t B_\parallel^\varepsilon + \dfrac{1}{\varepsilon}\, \nabla_\perp \cdot v_\perp^\varepsilon = \mathcal{O}(1)\,, \qquad \partial_t B_\perp^\varepsilon = \mathcal{O}(1)\,. \end{cases} \tag{6}$$

According to the terminology of Schochet [45], the asymptotic regime is called:

- *slow* when the first-order time derivative of the solution remains bounded uniformly with respect to the small paramater ε (as $\varepsilon \to 0$). In view of (6), this means that

$$\nabla_\perp \cdot v_\perp^\varepsilon = \mathcal{O}(\varepsilon)\,, \qquad \nabla_\perp (\wp^\epsilon + B_\parallel^\varepsilon) = \mathcal{O}(\varepsilon)\,. \tag{7}$$

- *fast* when it is not slow. In this case, rapid oscillations with non-vanishing amplitudes can persist on a long time scale, preventing the convergence in a usual strong sense. Since the singular part involves the sole action of the operator $\nabla_\perp$, it induces a propagation which can only be achieved with respect to the perpendicular direction. Then, because Alfvén waves do not propagate in the directions orthogonal to the ambiant magnetic field (here $e_\parallel$), we are necessarily concerned with transverse fast magnetosonic waves. This claim is justified in Section 3.1, where the eigenmodes of the linear (singular) system (6) are investigated.

At time $t = 0$, we start with

$$\rho_{|t=0}^\varepsilon = \rho_0^\varepsilon\,, \qquad v_{|t=0}^\varepsilon = v_0^\varepsilon\,, \qquad B_{|t=0}^\varepsilon = B_0^\varepsilon\,. \tag{8}$$

In coherence with (4), we must impose $\nabla_\varepsilon \cdot B_0^\varepsilon = 0$. The initial data is said to be *prepared* when

$$\nabla_\perp \cdot v_{0\perp}^\varepsilon = \mathcal{O}(\varepsilon)\,, \qquad \nabla_\perp (b\, \varrho_0^\epsilon + B_{0\parallel}^\varepsilon) = \mathcal{O}(\varepsilon)\,. \tag{9}$$

At this stage, it should be noted that the structure of the penalized terms inside (6) and of the subsequent condition (7) are different from isotropic situations [25]: the fluid should be almost incompressible only in the perpendicular direction (the action of the operator $\nabla_\perp$ appears in place of the full gradient ∇); the components ϱ_0^ϵ and $B_{0\parallel}^\varepsilon$ must be approximately linked together. To our knowledge, the asymptotic study of systems like (3) has not yet been undertaken neither in a smooth context or for weak solutions.

From now on, we assume that the positive perpendicular and parallel shear viscosities $\mu_\perp^\varepsilon > 0$ and $\mu_\parallel^\varepsilon > 0$, as well as the positive bulk viscosity $\lambda^\varepsilon > 0$ are adjusted in such a way that

$$\mu_\perp^\varepsilon \longrightarrow \mu_\perp > 0\,, \quad \mu_\parallel^\varepsilon \longrightarrow \mu_\parallel > 0\,, \quad \lambda^\varepsilon \longrightarrow \lambda > 0\,, \quad \text{as} \quad \varepsilon \longrightarrow 0_+\,. \tag{10}$$

Similarly the positive perpendicular and parallel resistivities $\eta_\perp^\varepsilon > 0$ and $\eta_\parallel^\varepsilon > 0$ must satisfy

$$\eta_\perp^\varepsilon \longrightarrow \eta_\perp > 0\,, \quad \eta_\parallel^\varepsilon \longrightarrow \eta_\parallel > 0\,, \quad \text{as} \quad \varepsilon \longrightarrow 0_+\,. \tag{11}$$

The system (3) is equipped with a conserved energy. We mainly assume that the energy of the initial data $(\rho_0^\varepsilon, v_0^\varepsilon, B_0^\varepsilon)$ is bounded uniformly with respect to ε, see (27) and (36). We add technical conditions which are distinct when $\Omega = \mathbb{T}^3$ (Subsection 2.3) and when $\Omega = \mathbb{R}^3$ (Subsection 2.4) to guarantee that the difference $\rho_0^\varepsilon - 1$ vanishes in $L^\gamma_{\mathrm{loc}}(\Omega)$ when ε goes to zero. Then, up to a subsequence and at least in a weak sense (specified further), we have

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$$\varrho_0^\epsilon := \frac{\rho_0^\varepsilon - 1}{\varepsilon} \rightharpoonup \varrho_0 \,, \qquad v_0^\varepsilon \rightharpoonup v_0 \in L^2 \,, \qquad B_0^\varepsilon \rightharpoonup B_0 \in L^2 \,, \quad \text{as} \quad \varepsilon \longrightarrow 0_+ \,.$$

We will show (Theorems 1 and 2) that, always up to a subsequence, the difference $\rho^\varepsilon - 1$ vanishes strongly in $L^\infty_{\rm loc}(\mathbb{R}_+; L^\gamma_{\rm loc}(\Omega))$ and that, at least in a weak sense, we have[1]

$$\varrho^\epsilon := \frac{\rho^\varepsilon - 1}{\varepsilon} \rightharpoonup \varrho \,, \qquad v^\varepsilon \rightharpoonup v \,, \qquad B^\varepsilon \rightharpoonup B \,, \quad \text{as} \quad \varepsilon \longrightarrow 0_+ \,.$$

The next stage is to identify the limit (ϱ, v, B).

1.2. The RMHD model

The perpendicular component $B_\perp := {}^t(B_1, B_2) \in \mathbb{R}^2$ of B and the perpendicular component $v_\perp := {}^t(v_1, v_2) \in \mathbb{R}^2$ of v can be identified independently by solving the following nonlinear closed system

$$\begin{cases} \partial_t B_\perp - \partial_\parallel v_\perp + \nabla_\perp \cdot (B_\perp \otimes v_\perp - v_\perp \otimes B_\perp) - \eta_\perp \Delta_\perp B_\perp - \eta_\parallel \Delta_\parallel B_\perp = 0 \,, \\ \partial_t v_\perp - \partial_\parallel B_\perp + \nabla_\perp \cdot (v_\perp \otimes v_\perp - B_\perp \otimes B_\perp) + \nabla_\perp \pi - \mu_\perp \Delta_\perp v_\perp - \mu_\parallel \Delta_\parallel v_\perp = 0 \,, \end{cases} \tag{12}$$

together with the (transverse velocity) divergence-free condition

$$\nabla_\perp \cdot v_\perp = 0 \,, \tag{13}$$

and the initial data

$$(B_\perp, v_\perp)_{|t=0} = (\mathbb{P}_\perp B_{0\perp}, \mathbb{P}_\perp v_{0\perp}) \in L^2(\Omega; \mathbb{R}^4) \,, \tag{14}$$

where the projection $\mathbb{P}_\perp$ denotes the (two-dimensional) transverse Leray operator. Passing to the weak limit in $\nabla_\varepsilon \cdot B_0^\varepsilon = 0$, we can easily infer that $\nabla_\perp \cdot B_{0\perp} = 0$, and therefore $B_{0\perp} = \mathbb{P}_\perp B_{0\perp}$. The same applies concerning v_0^ε in the case of prepared data. For unprepared data v_0^ε, in general, we find that $v_{0\perp} \neq \mathbb{P}_\perp v_{0\perp}$. Still, we will show in Section 4.5 that the limit initial condition is $\mathbb{P}_\perp v_{0\perp}$ and not $v_{0\perp}$. This passage from $v_{0\perp}$ to $\mathbb{P}_\perp v_{0\perp}$ reveals the underlying presence of a time boundary layer (which may arise in the absence of preparation). In the second equation of (12), the pressure π plays the role of a Lagrange multiplier to ensure the transverse incompressibility of the flow. Since

$$\nabla_\perp \cdot (B_\perp \otimes v_\perp - v_\perp \otimes B_\perp) = {}^t\big(\partial_2(B_1\, v_2 - v_1\, B_2), -\partial_1(B_1\, v_2 - v_1\, B_2)\big) \,,$$

exploiting (13), we can assert that

$$\partial_t(\nabla_\perp \cdot B_\perp) - \eta_\perp \Delta_\perp(\nabla_\perp \cdot B_\perp) - \eta_\parallel \Delta_\parallel(\nabla_\perp \cdot B_\perp) = 0 \,.$$

It follows that the divergence-free condition on $B_{0\perp}$ is propagated. Retain that

$$\nabla_\perp \cdot B_\perp = 0 \,. \tag{15}$$

The existence of global-in-time weak solutions to (12)-(13)-(14) can be obtained from classical methods in [14,42]. Note that it can be deduced indirectly from the existence (for all $\varepsilon > 0$) of weak solutions to (3). Indeed, as will be seen, the rigorous justification of the passage to the limit ($\varepsilon \to 0$) in the system (3)

[1] In the periodic case, let $\overline{\rho_0^\varepsilon}$ be the constant (close to 1) defined by (28). When $\Omega = \mathbb{T}^3$, as stated in Theorem 1, the definition of ϱ^ϵ should be replaced by $\varrho^\epsilon := (\rho^\varepsilon - \overline{\rho_0^\varepsilon})/\varepsilon$.

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provides another way to construct global weak solutions of (12)-(13)-(14). Observe also that the system is linear in the parallel direction. Thus, these global existence results remain true even when $\eta_\parallel = 0$ and $\mu_\parallel = 0$.

Now, the treatment of the parallel components $(B_\parallel, v_\parallel)$ differs completely from what is done usually. This is due to the (unconventional) anisotropic context which forces to look at $(B^\varepsilon_\parallel, v^\varepsilon_\parallel)$ separately. On the one hand, $(B^\varepsilon_\parallel, v^\varepsilon_\parallel)$ appears as a leading order term, and therefore its weak limit $(B_\parallel, v_\parallel)$ must contribute to the main description of the flow. As such, it must be incorporated in the RMHD model. On the other hand, to some extent, $(B^\varepsilon_\parallel, v^\varepsilon_\parallel)$ is (partly) dealt in the equations as a second order term. It follows that the determination of its weak limit $(B_\parallel, v_\parallel)$ is decoupled from the one of $(B_\perp, v_\perp)$. In fact, knowing the content of $(B_\perp, v_\perp)$, with the constant $\mathbb{c} := 1 + (1/b) > 1$, we have access to $(B_\parallel, v_\parallel)$ through

$$\begin{cases} \mathbb{c}\big(\partial_t B_\parallel + (v_\perp \cdot \nabla_\perp) B_\parallel\big) - \partial_\parallel v_\parallel - (B_\perp \cdot \nabla_\perp) v_\parallel - \eta_\perp \Delta_\perp B_\parallel - \eta_\parallel \Delta_\parallel B_\parallel = 0\,, \\ \partial_t v_\parallel + (v_\perp \cdot \nabla_\perp) v_\parallel - \partial_\parallel B_\parallel - (B_\perp \cdot \nabla_\perp) B_\parallel - \mu_\perp \Delta_\perp v_\parallel - \mu_\parallel \Delta_\parallel v_\parallel = 0\,, \end{cases} \tag{16}$$

and the initial data

$$(B_\parallel, v_\parallel)_{|t=0} = (\mathbb{B}_{0\parallel}, v_{0\parallel})\,, \qquad \mathbb{B}_{0\parallel} := (B_{0\parallel} - \varrho_0)/\mathbb{c}\,. \tag{17}$$

This is the viscous version of a symmetric linear system involving the known variable coefficients $v_\perp$ and $B_\perp$. For smooth data $v_\perp$ and $B_\perp$, the global existence is obvious. Moreover, due to (13) and (15), usual energy estimates concerning $(B_\parallel, v_\parallel)$ do apply without consuming any regularity on $v_\perp$ and $B_\perp$. It follows that global solutions do exist even when the coefficients $v_\perp$ and $B_\perp$ are issued from the weak solution $(v_\perp, B_\perp)$ in $L^\infty_{\text{loc}}(\mathbb{R}_+; L^2(\Omega)) \cap L^2_{\text{loc}}(\mathbb{R}_+, \dot{H}^1(\Omega))$ to (12).

Given $\varepsilon > 0$, the fluid is slightly compressible since $\rho^\varepsilon = 1 + \varepsilon\, \varrho^\varepsilon$ with $\varrho^\varepsilon \sim \varrho$. The expression ϱ^ε (and its weak limit ϱ) plays at the level of (3) the part of a one-order corrector which keeps track of the original compressibility. Now, looking at (6), it acts in the equations with the same order as the components $v^\varepsilon_\perp$ and $B^\varepsilon_\parallel$. It is therefore reasonable to find a link between ϱ, $v_\perp$ and $B_\parallel$. In view of the second relation inside (7), we can already infer that $\mathbb{p} + B_\parallel = 0$, where $\mathbb{p} := b\,\varrho$ is the weak limit of $\mathbb{p}^\varepsilon$. By this way, $B_\parallel$ acquires asymptotically the status of a pressure which can serve to measure some compressibility in the parallel direction. For prepared data, that is when $B_{0\parallel} + b\varrho_0 = 0$, we start with $\mathbb{B}_{0\parallel} = B_{0\parallel}$. Otherwise, for general unprepared data (which is our framework), we find that $\mathbb{B}_{0\parallel} \not\equiv B_{0\parallel}$, see Subsection 4.7. Again, this is the hallmark of a boundary layer occuring at time $t = 0$ concerning the component $B^\varepsilon_\parallel$.

1.3. *Global overview*

This paper is devoted to the rigorous justification of the convergence of the global weak solutions to (3) to those of (12)-(17). As already mentioned, the nature of the singular limit depends on many factors.

1.3.1. *Preceding results*

The hyperbolic version of system (3), which is obtained by removing viscosities and resistivities, falls into the framework of the theory of singular limits of quasilinear hyperbolic systems with large parameters. This approach is restricted to smooth solutions (say H^s with s large enough). It was originally developed by Klainerman and Majda [30,31,35]. In these circumstances, retain that:

- In the smooth prepared setting, as a corollary of Theorem 3 in [30] (see also §2.1 & §2.4 in [35], and references [43,44]), a convergence result does exist [21] concerning (3) without diffusion terms. It holds as long as the solution of the limit equations remains smooth. In a related framework, namely with a strong constant magnetic field but without spatial anisotropy, the authors of [25] study the singular limit of the local-in-time smooth solution of the ideal MHD on a bounded domain, with convenient boundary conditions and prepared initial data. It turns out that the limit system in [25] is essentially two-dimensional since the spatial variable x_3 plays the role of a label (no differential nor integral operator with respect to x_3). In [2], this method is successfully applied to the (more complex) XMHD system. Note also that we can appeal to Theorem 4 in [30] (dealing with the diffusive version of Theorem 3 in [30]; see also Theorem 4.1 in [46]) to justify the strong convergence on a fixed time interval of smooth prepared solutions to (3) to those of (12)-(17).

- In the smooth unprepared setting, one possible strategy [45] is first to exhibit a smooth (in its arguments) limit profile with a double number of variables (one set representing slow variations, the other set fast ones) satisfying an appropriate limit equation (called the modulation equation). Second, it is to prove that the smooth solution of the original system converges in a strong sense on a uniform time interval to this profile evaluated at the slow and fast variables.

Weak solutions can also be considered, provided that parabolic contributions are incorporated. This allows to relax the regularity conditions, to reach all times, and therefore to reinforce the universality of reduced models. A way to make progress in this direction has been initiated in [34] which (for unprepared data) exploits the unitary group method [20,45] and compactness arguments to construct a filtered profile for the irrotational part of the velocity field. From this filtered profile, the authors of [34] construct a sequence of approximations to the limit solution. Then, they exploit this sequence to pass to the limit in the nonlinear terms. By doing so, they observe that the solenoidal part of the velocity field inherits a strong convergence, while only weak convergence results are available concerning the irrotational part.

The discussion is very sensitive to the type of domain: $\mathbb{T}^3$ or $\mathbb{R}^3$. In the case of the whole space, the proof of [34] has been simplified in [11] by using Strichartz estimates [17,27]. This allows to improve the convergence result of the irrotational part of the velocity field, which is precisely the part containing the rapid oscillating acoustic waves. Indeed, the authors of [11] remark that this irrotational part satisfies a linear (isotropic) wave equation. From there, due to dispersive effects (in all spatial variables), it must asymptotically vanish in a strong sense.

1.3.2. The anisotropic complications

We clarify here the important unsolved specificities induced by the implementation of distinct spatial scales. In the smooth prepared context, new problems already arise. For instance, as observed in [2], the anisotropy can preclude obtaining a complete WKB expansion. Even in the smooth (prepared or not) case, the particularities related to the asymptotic study of (3) have not yet been explored. Inspired by [34], our aim is to go directly to weak solutions. We consider viscous and resistive situations vs. (almost) hyperbolic; global weak solutions vs. local strong solutions; L^p and periodic solutions vs. Sobolev solutions; and general data vs. prepared data. In so doing, the smooth strategies do not help. The good benchmark is [34]. But MHD equations are quite different from compressible fluid equations [34]. And thus, the discussion must be adapted to cover the magnetic effects. There are many important challenges to elucidate, especially:

- The unitary group method involves the quantities $\mathbb{Q}_\perp v_\perp^\varepsilon$ and $b\,\varrho^\varepsilon + B_\parallel^\varepsilon$. It allows to filter out fast oscillating magnetosonic waves propagating in the transverse directions in ways that have not yet been investigated (even in the smooth context). Note in particular that $b\,\varrho + B_\parallel = 0$, instead of simply $\varrho = 0$ in [34].
- The nonlinear expressions involving B^ε are, of course, absent in [34].
- Even the tensor product $\rho^\varepsilon v^\varepsilon \otimes v^\varepsilon$ must be dealt differently. Indeed, in our setting, both $v_\parallel^\varepsilon$ and $\mathbb{P}_\perp v_\perp^\varepsilon$ are left aside by the filtering. Other arguments must be introduced to understand what happens at the level of $v_\parallel^\varepsilon$ and $B_\parallel^\varepsilon$, that is how to recover (16). To deal with this issue, we exploit particular cancellations provided by the structure of system (3) that we combine with some compactness results developed in [33]

in order to prove the existence of weak solutions to the compressible Navier–Stokes equations. Indeed, in particular from (6), we observe that $\partial_t(B_\parallel^\varepsilon - \varrho^\varepsilon) = \mathcal{O}(1)$. Then, the quantity $(B_\parallel^\varepsilon - \varrho^\varepsilon)$ will yield the right unknown to prove the limit equation for $B_\parallel$. Moreover, in the case of the whole space, in contrast to [11], we cannot exploit (isotropic) Strichartz estimates to obtain the strong convergence of $\mathbb{Q}_\perp v_\perp^\varepsilon$, since the resulting wave equation is posed only in the perpendicular spatial variables $x_\perp$, the parallel spatial variable $x_\parallel$ being seen as a continuous label. Therefore, a natural and interesting open question arises: could only transverse dispersive effects (and thus some kind of anisotropic Strichartz's estimates) be used to show that $\mathbb{Q}_\perp v_\perp^\varepsilon$ vanishes strongly? This question will be addressed in further work.

1.3.3. Plan of the work

The paper is organized as follows. In Section 2, we state our main results. In Section 3, we come back to the physical motivations and to the origin of our anisotropic scaling. In Sections 4 and 5, we prove the convergence of the compressible MHD equations (3)-(4) to the RMHD equations (12)-(13)-(16). We start in Section 4 with the case of a periodic domain. Then, in Section 5, we perform this investigation in the whole space. Finally, in Appendix A, we recall functional analysis results which are exploited throughout the paper.

2. Main results

In Subsection 2.1, we specify some notations. In Subsection 2.2, we recall the notion of weak solutions. In Subsection 2.3, we state our main results in the case of $\mathbb{T}^3$. In Subsection 2.4, we do the same for $\mathbb{R}^3$.

2.1. Notation

Let Ω be either the periodic domain $\mathbb{T}^3$ or the whole space $\mathbb{R}^3$. For $s \in \mathbb{R}$ and $1 \le p \le \infty$, we shall use the standard non-homogeneous Sobolev spaces

$$W^{s,p}(\Omega) = (\mathrm{I} - \Delta)^{-s/2} L^p(\Omega)\,, \qquad H^s(\Omega) = W^{s,2}(\Omega)\,,$$

and their homogeneous versions

$$\dot{W}^{s,p}(\Omega) = (-\Delta)^{-s/2} L^p(\Omega)\,, \qquad \dot{H}^s(\Omega) = \dot{W}^{s,2}(\Omega)\,.$$

We introduce the transverse Leray projection operator $\mathbb{P}_\perp : W^{s,p}(\Omega;\mathbb{R}^2) \to W^{s,p}(\Omega;\mathbb{R}^2)$ onto vector fields which are divergence-free in the perpendicular direction,

$$v_\perp = \mathbb{P}_\perp v_\perp + \mathbb{Q}_\perp v_\perp\,, \qquad \nabla_\perp \cdot (\mathbb{P}_\perp v_\perp) = 0\,, \qquad \nabla_\perp \times (\mathbb{Q}_\perp v_\perp) = 0\,, \qquad \forall v_\perp \in L^2(\Omega;\mathbb{R}^2)\,,$$

where $\nabla_\perp \times v_\perp := \partial_1 v_2 - \partial_2 v_1$. From the Mikhlin–Hörmander Fourier multipliers theorem, the operators $\mathbb{P}_\perp$ and $\mathbb{Q}_\perp$ are continuous maps from the Sobolev space $W^{s,p}(\Omega;\mathbb{R}^2)$ into itself for $s \in \mathbb{R}$ and $1 < p < \infty$. In addition [34], for all $\delta > 0$, we have the following continuous embedding $\mathbb{P}_\perp(L^1(\Omega)) \hookrightarrow W^{-\delta,1}(\Omega)$. This embedding can be justified simply by observing that on the one hand the operators $\mathbb{P}_\perp$ and $\mathbb{Q}_\perp$ are continuous maps from $L^1(\Omega;\mathbb{R}^2)$ into the Lorentz space $L^{1,\infty}(\Omega)$ (or weak $L^1(\Omega)$; see, e.g., Theorem 5.3.3 in [19]) and on the other hand the continuous embedding $L^{1,\infty}(\Omega) \hookrightarrow W^{-\delta,1}(\Omega)$ holds. We denote by $\mathscr{C}(0,T;L^p_{\text{weak}}(\Omega))$, the space of functions which are continuous with respect to $t \in [0,T]$, with values in $L^p(\Omega)$, with the weak topology. Moreover, we introduce the differential operator $D_\varepsilon \equiv {}^t\nabla_\varepsilon$. The scalar product between two matrices M_1 and M_2 is defined as $M_1 : M_2 = \sum_{ij} M_{1ij} M_{2ij}$. Moreover, given two vectors $B_\perp \in \mathbb{R}^2$ and $v_\perp \in \mathbb{R}^2$, we adopt below the convention $B_\perp \times v_\perp := B_1 v_2 - B_2 v_1 \in \mathbb{R}$.

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2.2. Weak solutions

Weak solutions of RMHD equations will be recovered by passing to the limit ($\varepsilon \to 0_+$) in the weak formulation associated with (3)-(4). It is therefore important to specify what is meant by a weak solution to (3)-(4) and to (12)-(13)-(16) when $\Omega = \mathbb{T}^3$ and when $\Omega = \mathbb{R}^3$. Given initial data as in (8), with

$$\rho_0^\varepsilon \in L^1_{\rm loc}(\Omega)\,, \qquad v_0^\varepsilon\,,\, B_0^\varepsilon,\, \sqrt{\rho_0^\varepsilon} v_0^\varepsilon \in L^2(\Omega)\,, \qquad \nabla_\varepsilon \cdot B_0^\varepsilon = 0 \;\text{ in }\; \mathcal{D}'(\Omega)\,, \tag{18}$$

a triplet $(\rho^\varepsilon, v^\varepsilon, B^\varepsilon)$ satisfying

$$\rho^\varepsilon \in L^\infty_{\rm loc}(\mathbb{R}_+; L^\gamma_{\rm loc}(\Omega))\,, \qquad (v^\varepsilon, B^\varepsilon, \sqrt{\rho^\varepsilon} v^\varepsilon) \in L^\infty_{\rm loc}(\mathbb{R}_+; L^2(\Omega))\,, \tag{19}$$

is said to be a weak solution of (3)-(4) if for all $\psi = {}^t({}^t\psi_\perp, \psi_\parallel) \in \mathscr{C}^\infty_c(\mathbb{R}_+ \times \Omega; \mathbb{R}^3)$ and for all $\varphi \in \mathscr{C}^\infty_c(\mathbb{R}_+ \times \Omega; \mathbb{R})$ with $p^\varepsilon = a\,(\rho^\varepsilon)^\gamma$, we have

$$\int_\Omega dx \rho_0^\varepsilon \varphi(0) + \int_0^\infty dt \int_\Omega dx\, \rho^\varepsilon \big(\partial_t \varphi + v^\varepsilon \cdot \nabla_\varepsilon \varphi\big) = 0\,, \tag{20}$$

$$\begin{aligned}
&\int_\Omega dx\, \rho_0^\varepsilon v_{0\perp}^\varepsilon \cdot \psi_\perp(0) + \int_0^\infty dt \int_\Omega dx \Big(\rho^\varepsilon v_\perp^\varepsilon \cdot \partial_t \psi_\perp + \big(\rho^\varepsilon v_\perp^\varepsilon \otimes v^\varepsilon - B_\perp^\varepsilon \otimes B^\varepsilon\big) : D_\varepsilon \psi_\perp \\
&\quad + \Big\{ \frac{1}{\varepsilon^2} p^\varepsilon + \frac{B_\parallel^\varepsilon}{\varepsilon} + \frac{|B^\varepsilon|^2}{2} \Big\} \nabla_\perp \cdot \psi_\perp - B_\perp^\varepsilon \cdot \partial_\parallel \psi_\perp + \mu_\perp v_\perp^\varepsilon \cdot \Delta_\perp \psi_\perp + \mu_\parallel v_\perp^\varepsilon \cdot \Delta_\parallel \psi_\perp \Big) = 0\,,
\end{aligned} \tag{21}$$

$$\begin{aligned}
&\int_\Omega dx\, \rho_0^\varepsilon v_{0\parallel}^\varepsilon \psi_\parallel(0) + \int_0^\infty dt \int_\Omega dx \Big(\rho^\varepsilon v_\parallel^\varepsilon \partial_t \psi_\parallel + \big(\rho^\varepsilon v_\parallel^\varepsilon \otimes v^\varepsilon - B_\parallel^\varepsilon \otimes B^\varepsilon\big) : D_\varepsilon \psi_\parallel \\
&\quad + \Big\{ \frac{1}{\varepsilon} p^\varepsilon + \varepsilon \frac{|B^\varepsilon|^2}{2} \Big\} \partial_\parallel \psi_\parallel + \mu_\perp v_\parallel^\varepsilon \Delta_\perp \psi_\parallel + \mu_\parallel v_\parallel^\varepsilon \Delta_\parallel \psi_\parallel + \varepsilon \lambda^\varepsilon v^\varepsilon \cdot \nabla_\varepsilon (\partial_\parallel \psi_\parallel) \Big) = 0\,,
\end{aligned} \tag{22}$$

$$\begin{aligned}
&\int_\Omega dx\, B_{0\perp}^\varepsilon \cdot \psi_\perp(0) + \int_0^\infty dt \int_\Omega dx \Big(B_\perp^\varepsilon \cdot \partial_t \psi_\perp - v_\perp^\varepsilon \cdot \partial_\parallel \psi_\perp - (B_\perp^\varepsilon \times v_\perp^\varepsilon) \nabla_\perp \times \psi_\perp \\
&\quad + \varepsilon v_\parallel^\varepsilon B_\perp^\varepsilon \cdot \partial_\parallel \psi_\perp - \varepsilon B_\parallel^\varepsilon v_\perp^\varepsilon \cdot \partial_\parallel \psi_\perp + \eta_\perp B_\perp^\varepsilon \cdot \Delta_\perp \psi_\perp + \eta_\parallel B_\perp^\varepsilon \cdot \Delta_\parallel \psi_\perp \Big) = 0\,,
\end{aligned} \tag{23}$$

$$\begin{aligned}
&\int_\Omega dx\, B_{0\parallel}^\varepsilon \psi_\parallel(0) + \int_0^\infty dt \int_\Omega dx \Big(B_\parallel^\varepsilon \partial_t \psi_\parallel + \frac{1}{\varepsilon} v_\perp^\varepsilon \cdot \nabla_\perp \psi_\parallel \\
&\quad - v_\parallel^\varepsilon B_\perp^\varepsilon \cdot \nabla_\perp \psi_\parallel + B_\parallel^\varepsilon v_\perp^\varepsilon \cdot \nabla_\perp \psi_\parallel + \eta_\perp B_\parallel^\varepsilon \Delta_\perp \psi_\parallel + \eta_\parallel B_\parallel^\varepsilon \Delta_\parallel \psi_\parallel \Big) = 0\,,
\end{aligned} \tag{24}$$

$$\int_\Omega dx\, B^\varepsilon(t) \cdot \nabla_\varepsilon \varphi(t) = 0\,, \quad \forall t \in \mathbb{R}_+\,. \tag{25}$$

The notion of weak solution to (12)-(13) is obtained by testing (12) against all $\psi_\perp \in \mathscr{C}^\infty_c(\mathbb{R}_+ \times \Omega; \mathbb{R}^2)$ which are such that $\nabla_\perp \cdot \psi_\perp = 0$. Concerning (16), it suffices to select scalar test functions.

2.3. The periodic case

This is when $\Omega = \mathbb{T}^3$. The functional framework is based on [23,34]. Select initial data satisfying

$$\begin{cases} \rho^\varepsilon_{|t=0} = \rho^\varepsilon_0 \in L^\gamma(\mathbb{T}^3)\,, \qquad \rho^\varepsilon_0 \geq 0\,, \qquad \sqrt{\rho^\varepsilon_0} v^\varepsilon_0 \in L^2(\mathbb{T}^3)\,, \\ (\rho^\varepsilon v^\varepsilon)_{|t=0} = m^\varepsilon_0 = \begin{Bmatrix} \rho^\varepsilon_0 v^\varepsilon_0 & \text{if } \rho^\varepsilon_0 \neq 0 \\ 0 & \text{if } \rho^\varepsilon_0 = 0 \end{Bmatrix} \in L^{2\gamma/(\gamma+1)}(\mathbb{T}^3)\,, \\ B^\varepsilon_{|t=0} = B^\varepsilon_0 \in L^2(\mathbb{T}^3)\,, \qquad \nabla_\varepsilon \cdot B^\varepsilon_0 = 0\,, \qquad \int_{\mathbb{T}^3} dx\, B^\varepsilon_0 = 0\,. \end{cases} \tag{26}$$

We assume that these regularity assumptions are uniform with respect to ε. Furthermore, given a constant C_0 (not depending on ε), we impose

$$\frac{1}{2} \int_{\mathbb{T}^3} dx\, \big(\rho^\varepsilon_0 |v^\varepsilon_0|^2 + |B^\varepsilon_0|^2\big) + \frac{a}{\varepsilon^2(\gamma-1)} \int_{\mathbb{T}^3} dx\, \big((\rho^\varepsilon_0)^\gamma - \gamma \rho^\varepsilon_0 (\overline{\rho^\varepsilon_0})^{\gamma-1} + (\gamma-1)(\overline{\rho^\varepsilon_0})^\gamma\big) \leq C_0\,. \tag{27}$$

This is completed by

$$\overline{\rho^\varepsilon_0} := \frac{1}{|\mathbb{T}^3|} \int_{\mathbb{T}^3} dx\, \rho^\varepsilon_0 \longrightarrow 1\,, \quad \text{as } \varepsilon \longrightarrow 0_+\,. \tag{28}$$

This bound gives access to weak compactness. Modulo the extraction of subsequences (which are not specified), we can say that $\sqrt{\rho^\varepsilon_0} v^\varepsilon_0$ and B^ε_0 converge weakly in $L^2(\mathbb{T}^3)$ to u_0 and B_0 respectively. From (27), some information on ρ^ε_0 and $\varrho^\varepsilon_0 := (\rho^\varepsilon_0 - \overline{\varrho^\varepsilon_0})/\varepsilon$ can also be extracted. We will first show (see the proof of Lemma 1) that $\rho^\varepsilon_0 \to 1$ in $L^\gamma(\mathbb{T}^3)$−strong. Then, we will see that $\varrho^\varepsilon_0 \rightharpoonup \varrho_0$ in $L^\kappa(\mathbb{T}^3)$−weak for $\kappa := \min\{2, \gamma\}$. Using $\sqrt{\rho^\varepsilon_0} v^\varepsilon_0 \rightharpoonup u_0$ in $L^2(\mathbb{T}^3)$−weak and $\rho^\varepsilon_0 \to 1$ in $L^\gamma(\mathbb{T}^3)$−strong, we obtain $\rho^\varepsilon_0 v^\varepsilon_0 \rightharpoonup u_0 = v_0$ in $L^{2\gamma/(\gamma+1)}(\mathbb{T}^3)$−weak.

As soon as $\gamma > 3/2$, the contribution [23] furnishes a weak solution to (3)-(4) with

$$\begin{cases} \rho^\varepsilon \in L^\infty_{\text{loc}}\big(\mathbb{R}_+; L^\gamma(\mathbb{T}^3)\big)\,, \quad v^\varepsilon \in L^2_{\text{loc}}\big(\mathbb{R}_+; H^1(\mathbb{T}^3)\big)\,, \\ \sqrt{\rho^\varepsilon} v^\varepsilon \in L^\infty_{\text{loc}}\big(\mathbb{R}_+; L^2(\mathbb{T}^3)\big)\,, \quad \rho^\varepsilon v^\varepsilon \in L^\infty_{\text{loc}}\big(\mathbb{R}_+; L^{2\gamma/(\gamma+1)}(\mathbb{T}^3)\big) \cap \mathscr{C}_{\text{loc}}\big(\mathbb{R}_+; L^{2\gamma/(\gamma+1)}_{\text{weak}}(\mathbb{T}^3)\big)\,, \\ B^\varepsilon \in L^\infty_{\text{loc}}\big(\mathbb{R}_+; L^2(\mathbb{T}^3)\big) \cap \mathscr{C}_{\text{loc}}\big(\mathbb{R}_+; L^2_{\text{weak}}(\mathbb{T}^3)\big) \cap L^2_{\text{loc}}\big(\mathbb{R}_+; H^1(\mathbb{T}^3)\big)\,, \qquad \int_{\mathbb{T}^3} dx\, B^\varepsilon = 0\,. \end{cases} \tag{29}$$

The mass is conserved

$$\overline{\rho^\varepsilon} := \frac{1}{|\mathbb{T}^3|} \int_{\mathbb{T}^3} dx\, \rho^\varepsilon = \overline{\rho^\varepsilon_0}\,,$$

and thus, using (28), we deduce that $\overline{\rho^\varepsilon} \to 1$, as $\varepsilon \to 0_+$. Moreover, we have two energy inequalities

$$\mathbb{E}^\varepsilon(t) + \int_0^t ds\, \mathbb{D}^\varepsilon(s) \leq \mathbb{E}^\varepsilon_0\,, \quad \text{a.e. } t \in [0, +\infty)\,, \tag{30}$$

with $\mathbb{E} \in \{\mathcal{E}_1, \mathcal{E}_2\}$, where for $i = 1, 2$,

$$\mathcal{E}_i^\varepsilon(t) = \int_\Omega dx \left(\tfrac{1}{2}\rho^\varepsilon |v^\varepsilon|^2 + \tfrac{1}{2}|B^\varepsilon|^2 + \Pi_i(\rho^\varepsilon) \right), \quad \mathcal{E}_{i0}^\varepsilon = \int_\Omega dx \left(\tfrac{1}{2}\rho_0^\varepsilon |v_0^\varepsilon|^2 + \tfrac{1}{2}|B_0^\varepsilon|^2 + \Pi_i(\rho_0^\varepsilon) \right), \tag{31}$$

$$\mathbb{D}^\varepsilon = \int_\Omega dx \left(\mu_\perp^\varepsilon |\nabla_\perp v^\varepsilon|^2 + \mu_\parallel^\varepsilon |\partial_\parallel v^\varepsilon|^2 + \lambda^\varepsilon |\nabla_\varepsilon \cdot v^\varepsilon|^2 + \eta_\perp^\varepsilon |\nabla_\perp B^\varepsilon|^2 + \eta_\parallel^\varepsilon |\partial_\parallel B^\varepsilon|^2 \right), \tag{32}$$

and

$$\Pi_1(\rho^\varepsilon) = \frac{a}{\varepsilon^2(\gamma-1)}(\rho^\varepsilon)^\gamma, \quad \Pi_2(\rho^\varepsilon) = \frac{a}{\varepsilon^2(\gamma-1)}\left((\rho^\varepsilon)^\gamma - \gamma \rho^\varepsilon (\overline{\rho^\varepsilon})^{\gamma-1} + (\gamma-1)(\overline{\rho^\varepsilon})^\gamma \right). \tag{33}$$

For $i=1$, the inequality (30) is a consequence of straightforward calculation involving (3). Using the mass conservation, we can check that the inequality (30) for $i=2$ is equivalent to (30) for $i=1$. The case $i=2$ is introduced because it allows a better comparison of ρ^ε with $\overline{\rho^\varepsilon}$.

Theorem 1 *(Convergence of MHD to RMHD on a periodic domain). Assume $\Omega = \mathbb{T}^3$ and $\gamma > 3/2$. Consider a sequence $\{(\rho^\varepsilon, v^\varepsilon, B^\varepsilon)\}_{\varepsilon>0}$ of weak solutions to the compressible MHD system (3)-(4) with initial data $\{(\rho_0^\varepsilon, v_0^\varepsilon, B_0^\varepsilon)\}_{\varepsilon>0}$ as in (27). Let us set $\varrho^\varepsilon := (\rho^\varepsilon - \overline{\rho^\varepsilon})/\varepsilon$. Then, up to a subsequence, the family $\{(\rho^\varepsilon, \varrho^\varepsilon, v^\varepsilon, B^\varepsilon)\}_{\varepsilon>0}$ converges to $(1, \varrho, v, B)$ as indicated below*

$$\begin{aligned}
&\rho^\varepsilon \longrightarrow 1 \ \text{ in } \ L^\infty_{\text{loc}}(\mathbb{R}_+; L^\gamma(\mathbb{T}^3))-\text{strong}, \\
&\varrho^\varepsilon \rightharpoonup \varrho \ \text{ in } \ L^\infty_{\text{loc}}(\mathbb{R}_+; L^\kappa(\mathbb{T}^3))-\text{weak}-*, \quad \kappa = \min\{2,\gamma\}, \\
&\mathbb{P}_\perp v_\perp^\varepsilon \longrightarrow \mathbb{P}_\perp v_\perp = v_\perp \ \text{ in } \ L^2_{\text{loc}}(\mathbb{R}_+; L^p \cap H^s(\mathbb{T}^3))-\text{strong}, \quad 1 \le p < 6, \quad 0 \le s < 1, \\
&\mathbb{Q}_\perp v_\perp^\varepsilon \rightharpoonup 0 \ \text{ in } \ L^2_{\text{loc}}(\mathbb{R}_+; H^1(\mathbb{T}^3))-\text{weak}, \\
&v_\parallel^\varepsilon \rightharpoonup v_\parallel \ \text{ in } \ L^2_{\text{loc}}(\mathbb{R}_+; H^1(\mathbb{T}^3))-\text{weak}, \\
&B_\perp^\varepsilon \longrightarrow B_\perp \ \text{ in } \ L^r_{\text{loc}}(\mathbb{R}_+; L^2(\mathbb{T}^3))-\text{strong}, \quad 1 \le r < \infty, \\
&B_\parallel^\varepsilon \rightharpoonup B_\parallel \ \text{ in } \ L^2_{\text{loc}}(\mathbb{R}_+; H^1(\mathbb{T}^3))-\text{weak} \cap L^\infty_{\text{loc}}(\mathbb{R}_+; L^2(\mathbb{T}^3))-\text{weak}-*.
\end{aligned}$$

The limit point (v, B) is a weak solution to the RMHD equations (12)-(13)-(16) with initial data

$$(B_\perp, v_\perp)_{|t=0} = (B_{0\perp}, \mathbb{P}_\perp v_{0\perp}) \in L^2(\mathbb{T}^3), \qquad (B_\parallel, v_\parallel)_{|t=0} = (\mathbb{B}_{0\parallel}, v_{0\parallel}) \in L^2(\mathbb{T}^3),$$

where $\mathbb{B}_{0\parallel}$ is as in (17), and it satisfies the following regularity properties

$$B \in L^\infty_{\text{loc}}(\mathbb{R}_+; L^2(\mathbb{T}^3)) \cap L^2_{\text{loc}}(\mathbb{R}_+; H^1(\mathbb{T}^3)), \qquad v \in L^\infty_{\text{loc}}(\mathbb{R}_+; L^2(\mathbb{T}^3)) \cap L^2_{\text{loc}}(\mathbb{R}_+; H^1(\mathbb{T}^3)).$$

Moreover, the components ϱ and $B_\parallel$ are linked together by the relation $b\,\varrho + B_\parallel = 0$, for a.e. $(t,x) \in]0,+\infty[\times\mathbb{T}^3$.

2.4. The whole space case

This is when $\Omega = \mathbb{R}^3$. In order to define weak solutions in the whole space, we need to introduce the following special type of Orlicz spaces $L^p_q(\Omega)$ (see Appendix A of [33] for more details on these spaces),

$$L^p_q(\Omega) = \left\{ f \in L^1_{\text{loc}}(\Omega) \;\middle|\; f\mathbb{1}_{\{|f|\le\delta\}} \in L^q(\Omega), \ \ f\mathbb{1}_{\{|f|>\delta\}} \in L^p(\Omega), \ \delta > 0 \right\}, \tag{34}$$

where the function $\mathbb{1}_S$ denotes the indicator function of the set S. Obviously $L^p_p(\Omega) \equiv L^p(\Omega)$.

The functional framework is based on [23,34]. Select initial data satisfying

$$\begin{cases} \rho^\varepsilon_{|t=0} = \rho^\varepsilon_0 \in L^1_{\rm loc}(\mathbb{R}^3)\,, \quad \rho^\varepsilon_0 - 1 \in L^\gamma_2(\mathbb{R}^3)\,, \quad \gamma > 3/2\,, \quad \rho^\varepsilon_0 \geq 0\,, \quad \sqrt{\rho^\varepsilon_0}\, v^\varepsilon_0 \in L^2(\mathbb{R}^3)\,, \\ (\rho^\varepsilon v^\varepsilon)_{|t=0} = m^\varepsilon_0 = \begin{cases} \rho^\varepsilon_0 v^\varepsilon_0 & \text{if } \ \rho^\varepsilon_0 \neq 0 \\ 0 & \text{if } \ \rho^\varepsilon_0 = 0 \end{cases} \in L^1_{\rm loc}(\mathbb{R}^3)\,, \\ B^\varepsilon_{|t=0} = B^\varepsilon_0 \in L^2(\mathbb{R}^3)\,, \qquad \nabla_\varepsilon \cdot B^\varepsilon_0 = 0\,, \qquad \int_{\mathbb{R}^3} dx\, B^\varepsilon_0 = 0\,, \\ \rho^\varepsilon_0 \longrightarrow 1\,, \quad v^\varepsilon_0 \longrightarrow 0\,, \quad B^\varepsilon_0 \longrightarrow 0\,, \quad \text{as } |x| \longrightarrow \infty\,. \end{cases} \tag{35}$$

We assume that these regularity assumptions are uniform with respect to ε. Furthermore, given a constant C_0 (not depending on ε), we impose

$$\frac{1}{2}\int_{\mathbb{R}^3} dx\, \big(\rho^\varepsilon_0 |v^\varepsilon_0|^2 + |B^\varepsilon_0|^2\big) + \frac{a}{\varepsilon^2(\gamma-1)} \int_{\mathbb{R}^3} dx\, \big((\rho^\varepsilon_0)^\gamma - \gamma \rho^\varepsilon_0 + \gamma - 1\big) \leq C_0\,. \tag{36}$$

This bound gives access to weak compactness. Modulo the extraction of subsequences (which are not specified) we can say that $\sqrt{\rho^\varepsilon_0}\, v^\varepsilon_0$ and B^ε_0 converge weakly in $L^2(\mathbb{R}^3)$ to u_0 and B_0 respectively. From (36), we will show (see the proof of Lemma 6) the subsequent results. First, we will obtain (uniformly in ε) the bounds $\rho^\varepsilon_0 \in L^\gamma_{\rm loc}(\mathbb{R}^3)$, $\varrho^\varepsilon_0 \in L^\kappa_2 \cap L^\kappa_{\rm loc}(\mathbb{R}^3)$, with $\kappa = \min\{2,\gamma\}$, as well as $\rho^\varepsilon_0 v^\varepsilon_0 \in L^{2\gamma/(\gamma+1)}_{\rm loc}(\mathbb{R}^3)$. Second, we will obtain $\rho^\varepsilon_0 \to 1$ in $L^\gamma_2 \cap L^\gamma_{\rm loc}(\mathbb{R}^3)-$strong, and $\varrho^\varepsilon_0 \rightharpoonup \varrho_0$ in $L^\kappa_{\rm loc}(\mathbb{R}^3)-$weak. Moreover, using $\sqrt{\rho^\varepsilon_0}\, v^\varepsilon_0 \rightharpoonup u_0$ in $L^2(\mathbb{R}^3)-$weak and $\rho^\varepsilon_0 \to 1$ in $L^\gamma_{\rm loc}(\mathbb{R}^3)-$strong, we obtain $\rho^\varepsilon_0 v^\varepsilon_0 \rightharpoonup u_0 = v_0$ in $L^{2\gamma/(\gamma+1)}_{\rm loc}(\mathbb{R}^3)-$weak.

As soon as $\gamma > 3/2$, the contribution [23] furnishes a weak solution to (3)-(4) with

$$\begin{cases} \rho^\varepsilon \in L^\infty_{\rm loc}(\mathbb{R}_+; L^\gamma_{\rm loc}(\mathbb{R}^3))\,, \quad \rho^\varepsilon - 1 \in L^\infty_{\rm loc}(\mathbb{R}_+; L^\gamma_2(\mathbb{R}^3))\,, \quad \nabla v^\varepsilon \in L^2_{\rm loc}(\mathbb{R}_+; L^2(\mathbb{R}^3))\,, \\ \sqrt{\rho^\varepsilon}\, v^\varepsilon \in L^\infty_{\rm loc}(\mathbb{R}_+; L^2(\mathbb{R}^3))\,, \quad \rho^\varepsilon v^\varepsilon \in L^\infty_{\rm loc}\big(\mathbb{R}_+; L^{2\gamma/(\gamma+1)}_{\rm loc}(\mathbb{R}^3)\big) \cap \mathscr{C}_{\rm loc}\big(\mathbb{R}_+; L^{2\gamma/(\gamma+1)}_{\rm loc\, weak}(\mathbb{R}^3)\big)\,, \\ B^\varepsilon \in L^\infty_{\rm loc}(\mathbb{R}_+; L^2(\mathbb{R}^3)) \cap \mathscr{C}_{\rm loc}(\mathbb{R}_+; L^2_{\rm weak}(\mathbb{R}^3)) \cap L^2_{\rm loc}(\mathbb{R}_+; H^1(\mathbb{T}^3))\,, \quad \int_{\mathbb{T}^3} dx\, B^\varepsilon = 0\,, \\ \rho^\varepsilon \longrightarrow 1\,, \quad v^\varepsilon \longrightarrow 0\,, \quad B^\varepsilon \longrightarrow 0\,, \quad \text{as } |x| \longrightarrow \infty\,. \end{cases} \tag{37}$$

Moreover, we have the energy inequality (30) with $\mathbb{E}^\varepsilon = \mathcal{E}^\varepsilon_3$ given by the formula (31) and $\Pi_i = \Pi_3$, where Π_3 is defined by

$$\Pi_3(\rho^\varepsilon) = \frac{a}{\varepsilon^2(\gamma-1)}\big((\rho^\varepsilon)^\gamma - \gamma\rho^\varepsilon + \gamma - 1\big)\,. \tag{38}$$

This energy inequality is the consequence of straightforward calculation involving (3).

Theorem 2 *(Convergence of MHD to RMHD on the whole space). Assume $\Omega = \mathbb{R}^3$ and $\gamma > 3/2$. Consider $\{(\rho^\varepsilon, v^\varepsilon, B^\varepsilon)\}_{\varepsilon>0}$ a sequence of weak solutions to the compressible MHD system (3)-(4) with initial data $\{(\rho^\varepsilon_0, v^\varepsilon_0, B^\varepsilon_0)\}_{\varepsilon>0}$ as in (36). Let us set $\varrho^\varepsilon := (\rho^\varepsilon - 1)/\varepsilon$. Then, up to a subsequence, the family $\{(\rho^\varepsilon, \varrho^\varepsilon, v^\varepsilon, B^\varepsilon)\}_{\varepsilon>0}$ converge to $(1, \varrho, v, B)$ as indicated below*

$$\begin{aligned}
&\rho^\varepsilon \longrightarrow 1 \ \text{ in } \ L^\infty_{\rm loc}(\mathbb{R}_+; L^\gamma_2 \cap L^\gamma_{\rm loc} \cap H^{-\alpha}(\mathbb{R}^3))-\text{strong}\,, \quad \alpha \geq 1/2\,,\\
&\varrho^\varepsilon \rightharpoonup \varrho \ \text{ in } \ L^\infty_{\rm loc}(\mathbb{R}_+; L^\kappa_{\rm loc} \cap H^{-\alpha}(\mathbb{R}^3))-\text{weak}-*\,, \quad \kappa = \min\{2,\gamma\}\,, \quad \alpha \geq 1/2\,,\\
&\mathbb{P}_\perp v^\varepsilon_\perp \longrightarrow \mathbb{P}_\perp v_\perp = v_\perp \ \text{ in } \ L^2_{\rm loc}(\mathbb{R}_+; L^p_{\rm loc} \cap H^s_{\rm loc}(\mathbb{R}^3))-\text{strong}\,, \quad 1 \leq p < 6\,, \quad 0 \leq s < 1\,,\\
&\mathbb{Q}_\perp v^\varepsilon_\perp \rightharpoonup 0 \ \text{ in } \ L^2_{\rm loc}(\mathbb{R}_+; H^1(\mathbb{R}^3))-\text{weak}\,,\\
&v^\varepsilon_\parallel \rightharpoonup v_\parallel \ \text{ in } \ L^2_{\rm loc}(\mathbb{R}_+; H^1(\mathbb{R}^3))-\text{weak}\,,\\
&B^\varepsilon_\perp \longrightarrow B_\perp \ \text{ in } \ L^r_{\rm loc}(\mathbb{R}_+; L^2_{\rm loc}(\mathbb{R}^3))-\text{strong}\,, \quad 1 \leq r < \infty\,,\\
&B^\varepsilon_\parallel \rightharpoonup B_\parallel \ \text{ in } \ L^2_{\rm loc}(\mathbb{R}_+; H^1(\mathbb{R}^3))-\text{weak} \cap L^\infty_{\rm loc}(\mathbb{R}_+; L^2(\mathbb{R}^3))-\text{weak}-*\,.
\end{aligned}$$

The limit point (v, B) is a weak solution to the RMHD equations (12)-(13)-(16) with initial data

$$(B_\perp, v_\perp)_{|t=0} = (\mathbb{P}_\perp B_{0\perp}, \mathbb{P}_\perp v_{0\perp}) \in L^2(\mathbb{R}^3)\,, \qquad (B_\parallel, v_\parallel)_{|t=0} = (\mathbb{B}_{0\parallel}, v_{0\parallel}) \in L^2(\mathbb{R}^3)\,,$$

where $\mathbb{B}_{0\parallel}$ is as in (17), and it satisfies the following regularity properties

$$B \in L^\infty_{\rm loc}(\mathbb{R}_+; L^2(\mathbb{R}^3)) \cap L^2_{\rm loc}(\mathbb{R}_+; H^1(\mathbb{R}^3))\,, \qquad v \in L^\infty_{\rm loc}(\mathbb{R}_+; L^2(\mathbb{T}^3)) \cap L^2_{\rm loc}(\mathbb{R}_+; H^1(\mathbb{R}^3))\,.$$

Moreover, the components ϱ and $B_\parallel$ are linked together by the relation $b\,\varrho + B_\parallel = 0$, for a.e. $(t, x) \in]0, +\infty[\times\mathbb{R}^3$.

3. Physical motivations and scaling

The dimensional magnetohydrodynamic equations reads

$$\begin{cases}
\partial_{\rm t}\uprho + \nabla\cdot(\uprho \mathrm{v}) = 0\,,\\
\partial_{\rm t}(\uprho\mathrm{v}) + \nabla\cdot(\uprho\mathrm{v}\otimes\mathrm{v}) + \nabla\mathrm{p} + \mathrm{B}\times(\nabla\times\mathrm{B}) - \upmu_\perp\Delta_\perp\mathrm{v} - \upmu_\parallel\Delta_\parallel\mathrm{v} - \uplambda\nabla(\nabla\cdot\mathrm{v}) = 0\,,\\
\partial_{\rm t}\mathrm{B} + \nabla\times(\mathrm{B}\times\mathrm{v}) - \upeta_\perp\Delta_\perp\mathrm{B} - \upeta_\parallel\Delta_\parallel\mathrm{B} = 0\,,
\end{cases} \tag{39}$$

with the divergence-free condition $\nabla\cdot\mathrm{B} = 0$, and the barotropic closure $\mathrm{p} = \mathrm{p}(\uprho) = \mathfrak{a}\uprho^\gamma$, $\gamma > 1$. The triplet $(\uprho, \mathrm{v}, \mathrm{B}) = (\uprho, \mathrm{v}, \mathrm{B})(\mathrm{t}, \mathrm{x}_\perp, \mathrm{x}_\parallel) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3$ denotes respectively the dimensional fluid density, fluid velocity, and magnetic field. The variable t represents the dimensional time variable, while the two-dimensional (resp. one-dimensional) variable $\mathrm{x}_\perp$ (resp. $\mathrm{x}_\parallel$) represents the perpendicular (resp. parallel) dimensional space variable.

3.1. Large aspect ratio framework

Anisotopic plasmas with a strong background magnetic field are ubiquitus in astrophysical, space and fusion sciences. As an example, for fusion plasmas, the straight rectangular tokamak model involves a very long periodic column, whose section is a small periodic rectangle. The corresponding geometry and scalings are detailed carefully in [48]. Another example comes from various astrophysical plasmas such as the solar wind or the magnetosheath for which the underlying RMHD ordering is precisely described in [41]. In order to obtain the dimensionless MHD equations (3), we must first nondimensionalize equations (39), and then choose a scaling. Putting dimensions into the “bar” quantities, we define the dimensionless unknowns and variables as $\mathrm{t} = \bar{\mathrm{t}}\,t$, $\mathrm{x}_\perp = \bar{\mathrm{x}}_\perp\,x_\perp$, $\mathrm{x}_\parallel = \bar{\mathrm{x}}_\parallel\,x_\parallel$, $\upmu_\perp = \bar{\upmu}_\perp\,\mu_\perp$, $\upmu_\parallel = \bar{\upmu}_\parallel\,\mu_\parallel$, $\uplambda = \bar{\uplambda}\,\lambda$, $\upeta_\perp = \bar{\upeta}_\perp\,\eta_\perp$, $\upeta_\parallel = \bar{\upeta}_\parallel\,\eta_\parallel$, $\mathfrak{a} = \bar{\mathfrak{a}}\,a$, $\uprho = \bar{\uprho}\,\rho$, $\mathrm{v} = \bar{\mathrm{v}}\,v$, and $\mathrm{B} = \bar{\mathrm{B}}\,B$. From this, and the barotropic state law, we deduce the dimensionless pressure as $\mathrm{p} = \bar{\mathrm{p}}\,p$ with $\bar{\mathrm{p}} = \bar{\mathfrak{a}}\bar{\uprho}^\gamma$ and $p = a\rho^\gamma$. We also define important physical quantities such as the Alfvén velocity $v_A := \bar{\mathrm{B}}/\sqrt{\bar{\uprho}}$, the sound velocity $v_s := \sqrt{\gamma\bar{\mathrm{p}}/\bar{\uprho}}$, the parameter $\beta := \bar{\mathrm{p}}/|\bar{\mathrm{B}}|^2 = v_s^2/(\gamma v_A^2)$, the

Alfvén number $\varepsilon_A := \bar{\mathrm{v}}/v_A$ and the Mach number $\varepsilon_M := \bar{\mathrm{v}}/v_s = \varepsilon_A/(\sqrt{\gamma\beta})$. Here, we suppose that the parameter β is of order one, and thus we set $\beta = 1$. This configuration is called the *high* β ordering [49], and often appears in space plasmas [3,29,38]. In other situations, such as plasmas of tokamaks [13,24,26], the parameter β can be relatively small; this is the so-called *low* β regim. Indeed, since β measures the ratio of the fluid pressure to the magnetic pressure, a magnetically well-confined plasmas is achieved for low β. Since here we choose $\beta = 1$, we have $v_s = \sqrt{\gamma}v_A \simeq v_A$.

In order to understand which parts of the solution of the MHD equations (3) are eliminated in the reduced model (12)-(17), we now recall the different types of (linear) waves propagating in a plasma governed by the MHD equations (39). Dropping viscosities and resistivities terms, it is well-known [40] that the system (39) is hyperbolic, but not strictly hyperbolic since some eigenvalues may coincide. Linearizing the system (39) around the constant stationary solution $(\bar{\mathrm{\rho}}, 0, \bar{\mathrm{B}}\,\mathbb{b})$, where $\mathbb{b}$ is a unit vector, we obtain a linear system whose the Jacobian has real eigenvalues [18]. The set of MHD eigenvalues and associated waves can be splitted into three groups. Introducing the unit vector $\mathbb{n}$ as the direction of propagation of any wave, the sound speed $V_s := \sqrt{\gamma \mathrm{p}/\mathrm{\rho}} = \sqrt{a}v_s$ (with $\rho = 1$) and the Alfvén velocity $V_A := |\mathrm{B}|/\sqrt{\mathrm{\rho}} = v_A$ (with $|B| = 1$ and $\rho = 1$), these three groups are [18]:

- Fast magnetosonic waves:

$$\lambda_F^\pm = \pm\mathcal{C}_F\,, \quad \mathcal{C}_F^2 = \frac{1}{2}\Big(V_s^2 + V_A^2 + \sqrt{(V_s^2+V_A^2)^2 - 4V_s^2V_A^2(\mathbb{b}\cdot\mathbb{n})^2}\Big)\,.$$

- Alfvén waves:

$$\lambda_A^\pm = \pm\mathcal{C}_A\,, \quad \mathcal{C}_A^2 = V_A^2(\mathbb{b}\cdot\mathbb{n})^2\,.$$

- Slow magnetosonic waves:

$$\lambda_S^\pm = \pm\mathcal{C}_S\,, \quad \mathcal{C}_S^2 = \frac{1}{2}\Big(V_s^2 + V_A^2 - \sqrt{(V_s^2+V_A^2)^2 - 4V_s^2V_A^2(\mathbb{b}\cdot\mathbb{n})^2}\Big)\,.$$

Since here $\mathbb{b} := e_\parallel$, Alfvén waves cannot propagate in the perpendicular direction to $e_\parallel$. Indeed, it is well-known [18] that Alfvén waves propagate mainly along the direction $(\mathbb{b} := e_\parallel)$ of the ambiant magnetic field. For a wave propagating in the perpendicular direction to $e_\parallel$, we obtain $\lambda_F^\pm = \pm(V_s^2+V_A^2)^{1/2} \simeq \pm V_A \simeq \pm V_s$, whereas $\lambda_S^\pm = 0$. Note that in dimensionless variables we have $\lambda_F^\pm = \pm\sqrt{b+1}$, with $b = a\gamma$. Indeed, normalizing the velocity to the Alfvén velocity v_A and taking $\beta = 1$ in $\lambda_F^\pm = \pm v_A\sqrt{\beta a\gamma + 1}$, we obtain the desired result. In order to understand now the nature of the waves that are filtered out from the singular part of the linear system (6), we rewrite it in the fast time variable t to obtain

$$\partial_{\mathrm{t}}\varrho + \nabla_\perp\cdot v_\perp = 0\,, \qquad \partial_{\mathrm{t}}v_\perp + \nabla_\perp(b\varrho + B_\parallel) = 0\,, \qquad \partial_{\mathrm{t}}B_\parallel + \nabla_\perp\cdot v_\perp = 0\,.$$

With $U := {}^t(\varrho, v_1, v_2, B_\parallel)$, the previous system can be recast as $\partial_{\mathrm{t}}U + (A_1\partial_{x_1} + A_1\partial_{x_2})U = 0$, where the matrices A_i have constant coefficients depending on b. With $\mathbb{n}_\perp = {}^t(\mathbb{n}_1, \mathbb{n}_2)$ a unit vector in the perpendicular direction, the matrix $\mathcal{A} := \mathbb{n}_1A_1 + \mathbb{n}_2A_2$ is diagonalizable with the real eigenvalues $\lambda_0(\mathcal{A}) = 0$ (of multiplicity two), $\lambda_+(\mathcal{A}) = \sqrt{b+1}$, and $\lambda_-(\mathcal{A}) = -\sqrt{b+1}$. Then, the waves associated with the singular part of the linear system (6) are the transverse (linear) fast magnetosonic waves.

Therefore, here, we aim at filtering out the fast dynamics associated with the perpendicular fast magnetosonic waves, and keep the dynamics of waves which propagate at a speed slower than the perpendicular fast magnetosonic waves $\mathcal{C}_F \simeq v_A$. Defining the time $\mathrm{\tau}_\perp$ as the time needed by a fast magnetosonic waves to cross the device in the perpendicular direction, we then have $\mathrm{\tau}_\perp v_A = \bar{\mathrm{x}}_\perp$. Since we want to describe the dynamics on a time scale longer that $\mathrm{\tau}_\perp$, we set $\bar{\mathrm{t}} = \mathrm{\tau}_\perp/\varepsilon$, with $\varepsilon \ll 1$. This is equivalent to describe the dynamics of waves which propagate with velocity slower than $\mathcal{C}_F$ (or v_A). Hence, we have $\bar{\mathrm{v}} = \varepsilon v_A$, $\varepsilon_A = \varepsilon$, and $\varepsilon_M = \varepsilon/(\sqrt{\gamma\beta}) \simeq \varepsilon$. Moreover, we suppose a strong anisotropy between the perpendicular and parallel direction, that is $\bar{\mathrm{x}}_\perp/\bar{\mathrm{x}}_\parallel = \varepsilon$. In other words the dimensional gradient ∇_{x} becomes the anisotropic dimensionless gradient ∇_ε of (2). In addition, we suppose the presence of a strong constant background magnetic field in the parallel direction ($B = e_\parallel + \varepsilon B^\varepsilon$). Since ε is present in the resulting dimensionless system, the velocity field v and the density ρ will depend on ε, hence we set $v = v^\varepsilon$ and $\rho = \rho^\varepsilon$. Finally, it remains to choose some scalings with respect to the small parameter ε for the dimensionless viscosities and resistivities. We choose $\{\mu_\perp = \varepsilon\mu_\perp^\varepsilon,\ \mu_\parallel = \mu_\parallel^\varepsilon/\varepsilon,\ \lambda = \varepsilon\lambda^\varepsilon\}$, where viscosities $\{\mu_\perp^\varepsilon,\ \mu_\parallel^\varepsilon,\ \lambda^\varepsilon\}$ satisfy (10), and $\{\eta_\perp = \varepsilon\eta_\perp^\varepsilon,\ \eta_\parallel = \eta_\parallel^\varepsilon/\varepsilon\}$, where resistivities $\{\eta_\perp^\varepsilon,\ \eta_\parallel^\varepsilon\}$ satisfy (11).

All the above considerations allow us to pass from the dimensional MHD equations (39) to the dimensionless ones (3).

3.2. Nonlinear optics framework

Conducting fluids are traversed by electromagnetic waves, which can interact with the medium in various ways. These phenomena can be modeled by adjusting the dimensionless parameters to account for special regimes, and by incorporating (high frequency) oscillating source terms or equivalently (high frequency) oscillating initial data into the equations. Here, we choose viscosities and resistivities which accommodate the propagation of oscillating waves with wavelengths approximately ε. For this, we impose viscosities $\{\mathrm{\mu}_\perp = \varepsilon^2\mu_\perp^\varepsilon,\ \mathrm{\mu}_\parallel = \mu_\parallel^\varepsilon,\ \mathrm{\lambda} = \varepsilon^2\lambda^\varepsilon\}$, where dimensionless viscosities $\{\mu_\perp^\varepsilon,\ \mu_\parallel^\varepsilon,\ \lambda^\varepsilon\}$ satisfy (10), and resistivities $\{\mathrm{\eta}_\perp = \varepsilon^2\eta_\perp^\varepsilon,\ \mathrm{\eta}_\parallel = \eta_\parallel^\varepsilon\}$, where dimensionless resistivities $\{\eta_\perp^\varepsilon,\ \eta_\parallel^\varepsilon\}$ satisfy (11). We then look for solution like

$$\begin{aligned}\begin{pmatrix}\mathrm{\rho}/\bar{\mathrm{\rho}}\\ \mathrm{v}/\bar{\mathrm{v}}\\ \mathrm{B}/\bar{\mathrm{B}}\end{pmatrix}(\mathrm{t},\mathrm{x}_\perp,\mathrm{x}_\parallel) &= \begin{pmatrix}\rho^\varepsilon\big(\mathrm{t},\varepsilon^{-1}\,\mathrm{x}_\perp,\mathrm{x}_\parallel\big)\\ \varepsilon\, v^\varepsilon\big(\mathrm{t},\varepsilon^{-1}\,\mathrm{x}_\perp,\mathrm{x}_\parallel\big)\\ 1+\varepsilon\, B^\varepsilon\big(\mathrm{t},\varepsilon^{-1}\,\mathrm{x}_\perp,\mathrm{x}_\parallel\big)\end{pmatrix} = \begin{pmatrix}\overline{\rho^\varepsilon}\\ 0\\ 1\end{pmatrix} + \varepsilon\begin{pmatrix}\varrho^\varepsilon\\ v^\varepsilon\\ B^\varepsilon\end{pmatrix}\big(\mathrm{t},\varepsilon^{-1}\,\mathrm{x}_\perp,\mathrm{x}_\parallel\big)\\ &= \begin{pmatrix}\overline{\rho^\varepsilon}\\ 0\\ 1\end{pmatrix} + \varepsilon\begin{pmatrix}\varrho^\varepsilon\\ v^\varepsilon\\ B^\varepsilon\end{pmatrix}\big(t,x_\perp,x_\parallel\big)\,, \qquad (40)\end{aligned}$$

where $\varrho^\varepsilon = (\rho^\varepsilon - \overline{\rho^\varepsilon})/\varepsilon$. Plugging (40) into (39) leads to (3). Therefore, we investigate the dynamics of a magnetized plasma near a fixed large constant magnetic field where anisotropic oscillations in space can develop. The first term of the right-hand side of (40), which is of order of unity, is a stationary solution of (39). The second term of the right-hand side of (40) is the perturbation, which is of small amplitude ($\varepsilon \ll 1$) and of high frequency ($\varepsilon^{-1} \gg 1$). Such a framework belongs to the so called weakly nonlinear geometric optics regim.

4. Asymptotic analysis in a periodic domain

This section is devoted to the proof of Theorem 1. First, we obtain some weak compactness properties for the sequences, ϱ^ε, $B_\parallel^\varepsilon$, v^ε, $\rho^\varepsilon v^\varepsilon$, and $\mathbb{Q}_\perp v_\perp^\varepsilon$, and strong ones for the sequences ρ^ε, $B_\perp^\varepsilon$, $\mathbb{P}_\perp v_\perp^\varepsilon$, $\mathbb{P}_\perp(\rho^\varepsilon v_\perp^\varepsilon)$ and $(\rho^\varepsilon v^\varepsilon - v^\varepsilon)$. Using these compactness results, we justify the passage to the limit, in order, in the equations of ρ^ε, $\rho^\varepsilon v_\perp^\varepsilon$, $\rho^\varepsilon v_\parallel^\varepsilon$, $B_\perp^\varepsilon$, and $B_\parallel^\varepsilon$ (or equivalently ϱ^ε). For the equations of $\rho^\varepsilon v_\perp^\varepsilon$, we use the unitary group method, while for the equations of $\rho^\varepsilon v_\parallel^\varepsilon$, and $B_\parallel^\varepsilon$, we use some particular cancellations and a compactness argument (Lemma 14 of Appendix A).

4.1. Compactness of ρ^ε and ϱ^ε

Here, we aim at proving the following lemma.

Lemma 1. *Assume $\gamma > 3/2$. The sequences ρ^ε and $\varrho^\varepsilon := (\rho^\varepsilon - \overline{\rho^\varepsilon})/\varepsilon$ satisfy the following properties.*

$$\rho^\varepsilon \longrightarrow 1 \quad \text{in} \quad L^\infty_{\text{loc}}(\mathbb{R}_+; L^\gamma(\mathbb{T}^3))-\text{strong}\,,$$
$$\varrho^\varepsilon \rightharpoonup \varrho \quad \text{in} \quad L^\infty_{\text{loc}}(\mathbb{R}_+; L^\kappa(\mathbb{T}^3))-\text{weak}-*\,, \quad \kappa = \min\{2,\gamma\}\,.$$

Proof. Let us start with ρ^ε. From energy inequality (30)-(33) with the pressure term Π_1, we already know that $\Pi_1(\rho^\varepsilon)$ is bounded in $L^\infty_{\text{loc}}(\mathbb{R}_+; L^1(\mathbb{T}^3))$, uniformly with respect to ε. Thus, by weak compactness, we have $\rho^\varepsilon \to 1$ in $L^\infty_{\text{loc}}(\mathbb{R}_+; L^\gamma(\mathbb{T}^3))-\text{weak}-*$. In addition, since $\overline{\rho^\varepsilon} = \overline{\rho^\varepsilon_0} \to 1$ as $\varepsilon \to 0$, for ε small enough we have $\overline{\rho^\varepsilon} \in (1/2, 3/2)$. Then, using Lemma 11, we claim that there exists $\eta = \eta_\delta = \eta(\gamma,\delta) > 0$, such that for $|x - \overline{\rho^\varepsilon}| \geq \delta$ and $x \geq 0$, we have

$$x^\gamma - \gamma x \bar{x}^{\gamma-1} + (\gamma - 1)\bar{x}^\gamma \geq \eta_\delta \big|x - \overline{\rho^\varepsilon}\big|^\gamma. \tag{41}$$

Indeed, using Lemma 11 with $\bar{x} = \overline{\rho^\varepsilon} \in (1/2, 3/2)$, we obtain, $\eta = \nu_3$, for $1 < \gamma < 2$, and $3/2 < R < x$; $\eta = \delta^{2-\gamma}\nu_2$, for $1 < \gamma < 2$, and $x \leq R$; and $\eta = \nu_1 \sup_{x\in\mathbb{T}^3} |x - \overline{\rho^\varepsilon}|^{2-\gamma} > 0$, for $\gamma \geq 2$, and $x \geq 0$. Therefore, using (41), inequality $(a/2 + b/2)^\gamma \leq (a^\gamma + b^\gamma)/2$ (by convexity of $x \mapsto x^\gamma$), and energy inequality (30)-(33) with the pressure term Π_2, we obtain

$$\begin{aligned}\sup_{t\geq 0}\int_{\mathbb{T}^3} dx\, |\rho^\varepsilon - 1|^\gamma &\leq 2^{\gamma-1}\big|\mathbb{T}^3\big|\big|\overline{\rho^\varepsilon} - 1\big|^\gamma + 2^{\gamma-1}\sup_{t\geq 0}\Bigg\{\int_{|\rho^\varepsilon - \overline{\rho^\varepsilon}|\leq\delta} dx + \int_{|\rho^\varepsilon - \overline{\rho^\varepsilon}|>\delta} dx\Bigg\}\big|\rho^\varepsilon - \overline{\rho^\varepsilon}\big|^\gamma \\ &\leq 2^{\gamma-1}\Bigg\{\big|\mathbb{T}^3\big|\big|\overline{\rho^\varepsilon} - 1\big|^\gamma + \big|\mathbb{T}^3\big|\delta^\gamma + \frac{C_0\varepsilon^2}{\eta_\delta}\Bigg\}.\end{aligned}$$

In the previous estimate, taking first $\varepsilon \to 0$, and then $\delta \to 0$, lead to the convergence of ρ^ε as stated in Lemma 1. We continue with $\varrho^\varepsilon := (\rho^\varepsilon - \overline{\rho^\varepsilon})/\varepsilon$. Using Lemma 11 with $\bar{x} = \overline{\rho^\varepsilon} \in (1/2, 3/2)$ and $x = \rho^\varepsilon$, and using energy inequality (30)-(33) with the pressure term Π_2, there exists a constant C independent of ε such that

$$\begin{cases} \text{if} \quad \gamma \geq 2, \quad \|\varrho^\varepsilon\|_{L^\infty_{\text{loc}}(\mathbb{R}_+;L^2(\mathbb{T}^3))} \leq C\,, \\ \text{if} \quad \gamma < 2, \ \forall R \in \big(\frac{3}{2}, +\infty\big), \ \|\varrho^\varepsilon \mathbb{1}_{\rho^\varepsilon < R}\|_{L^\infty_{\text{loc}}(\mathbb{R}_+;L^2(\mathbb{T}^3))} \leq C\,, \ \|\varrho^\varepsilon \mathbb{1}_{\rho^\varepsilon \geq R}\|_{L^\infty_{\text{loc}}(\mathbb{R}_+;L^\gamma(\mathbb{T}^3))} \leq C\varepsilon^{(2/\gamma)-1}\,. \end{cases} \tag{42}$$

Using this last estimate, ϱ^ε is bounded, uniformly with respect to ε, in $L^\infty_{\text{loc}}(\mathbb{R}_+; L^\kappa(\mathbb{T}^3))$, with $\kappa = \min\{2,\gamma\}$. Hence, weak compactness leads to the convergence of ϱ^ε stated in Lemma 1. □

4.2. Compactness of B^ε

Here, we aim at proving the following lemma.

Lemma 2. *The sequence B^ε satisfies the following properties.*

$$B^\varepsilon \rightharpoonup B \quad \text{in} \quad L^2_{\text{loc}}(\mathbb{R}_+; L^6 \cap H^1(\mathbb{T}^3))-\text{weak} \cap L^\infty_{\text{loc}}(\mathbb{R}_+; L^2(\mathbb{T}^3))-\text{weak}-*\,,$$
$$\nabla_\varepsilon \cdot B^\varepsilon \rightharpoonup \nabla_\perp \cdot B_\perp = 0 \quad \text{in} \quad L^2_{\text{loc}}(\mathbb{R}_+; L^2(\mathbb{T}^3))-\text{weak}\,,$$
$$B^\varepsilon_\perp \longrightarrow B_\perp \quad \text{in} \quad L^r_{\text{loc}}(\mathbb{R}_+; L^2(\mathbb{T}^3))-\text{strong}\,, \quad 1 \leq r < +\infty\,.$$

Proof. The first limit of Lemma 2, comes on the one hand from weak compactness, and on the other hand from energy inequality (30)-(33) with the pressure term Π_2, and the continuous Sobolev embeddings

$H^1(\mathbb{T}^3) \hookrightarrow L^6(\mathbb{T}^3)$, which implies that B^ε is bounded in $L^2_{\text{loc}}(\mathbb{R}_+; H^1 \cap L^6(\mathbb{T}^3))$ and in $L^\infty_{\text{loc}}(\mathbb{R}_+; L^2(\mathbb{T}^3))$, uniformly with respect to ε.

We continue with the second assertion. Since B^ε is uniformly bounded in $L^2_{\text{loc}}(\mathbb{R}_+; H^1(\mathbb{T}^3))$ and $\nabla_\perp \cdot B^\varepsilon_\perp = -\varepsilon\partial_\parallel B^\varepsilon_\parallel$ in $\mathcal{D}'$, we obtain $\|\nabla_\perp \cdot B^\varepsilon_\perp\|_{L^2_{\text{loc}}(\mathbb{R}_+;L^2(\mathbb{T}^3))} \leq \varepsilon \|\partial_\parallel B^\varepsilon_\parallel\|_{L^2_{\text{loc}}(\mathbb{R}_+;L^2(\mathbb{T}^3))} \leq C_0\varepsilon$. On the one hand $\nabla_\perp \cdot B^\varepsilon_\perp$ is bounded in $L^2_{\text{loc}}(\mathbb{R}_+; L^2(\mathbb{T}^3))$ and goes to $\nabla_\perp \cdot B_\perp \in \mathcal{D}'$. On the other hand, it must vanish as $\varepsilon \to 0$. Hence, the second line.

For the third assertion, we apply Lemma 13 of Appendix A with $\mathfrak{B}_0 = H^1(\mathbb{T}^3)$, $\mathfrak{B} = L^2(\mathbb{T}^3)$, $\mathfrak{B}_1 = H^{-1}(\mathbb{T}^3)$, $p = r$, and $q = \infty$. To this end, we have to check the corresponding hypotheses. Below (and after when there is no possible ambiguity), *bounded* means "uniformly bounded with respect to ε".

- From energy inequality (30)-(33) and the continuous Sobolev embedding $H^1(\mathbb{T}^3) \hookrightarrow L^6(\mathbb{T}^3)$, we obtain that B^ε is bounded in $L^\infty_{\text{loc}}(\mathbb{R}_+; L^2(\mathbb{T}^3)) \cap L^2_{\text{loc}}(\mathbb{R}_+; H^1 \cap L^6(\mathbb{T}^3))$.
- Obviously, $B^\varepsilon_\perp$ is bounded in $L^\infty_{\text{loc}}(\mathbb{R}_+; L^2(\mathbb{T}^3)) \cap L^1_{\text{loc}}(\mathbb{R}_+; H^1(\mathbb{T}^3))$.
- The final step is to estimate $\partial_t B^\varepsilon_\perp$. We can exploit equation (1) to express this time derivative. Observe that, as can be seen at the level of (23), there is no singular term in ε. The Laplacian parts are clearly bounded in $L^1_{\text{loc}}(\mathbb{R}_+; H^{-1}(\mathbb{T}^3))$. Let us consider the products of components of B^ε and v^ε. We refer to (the proof of) Lemma 3 which guarantees that v^ε is bounded in $L^2_{\text{loc}}(\mathbb{R}_+; H^1 \cap L^6(\mathbb{T}^3))$. Hence, by Hölder inequality, these products are bounded in $L^1_{\text{loc}}(\mathbb{R}_+; L^3(\mathbb{T}^3))$. Since $L^3(\mathbb{T}^3) \hookrightarrow L^2(\mathbb{T}^3)$, after spatial derivation, we find as required a bound in $L^1_{\text{loc}}(\mathbb{R}_+; H^{-1}(\mathbb{T}^3))$ for $\partial_t B^\varepsilon_\perp$. □

4.3. Compactness of v^ε and $\rho^\varepsilon v^\varepsilon$

Here we aim at proving the following lemma.

Lemma 3. *Assume $\gamma > 3/2$. Let $\mathfrak{s} := \max\{1/2, 3/\gamma - 1\} \in [1/2, 1)$. The sequences v^ε and $\rho^\varepsilon v^\varepsilon$ satisfy the following properties.*

$$
\begin{aligned}
& v^\varepsilon \rightharpoonup v \quad \text{in} \quad L^2_{\text{loc}}(\mathbb{R}_+; L^6 \cap H^1(\mathbb{T}^3))-\text{weak}\,,\\
& \nabla_\varepsilon \cdot v^\varepsilon \rightharpoonup \nabla_\perp \cdot v_\perp = 0 \quad \text{in} \quad L^2_{\text{loc}}(\mathbb{R}_+; L^2(\mathbb{T}^3))-\text{weak}\,,\\
& \mathbb{P}_\perp v^\varepsilon_\perp \rightharpoonup \mathbb{P}_\perp v_\perp \quad \textit{and} \quad \mathbb{Q}_\perp v^\varepsilon_\perp \rightharpoonup \mathbb{Q}_\perp v_\perp = 0 \quad \text{in} \quad L^2_{\text{loc}}(\mathbb{R}_+; L^6 \cap H^1(\mathbb{T}^3))-\text{weak}\,,\\
& \mathbb{P}_\perp v^\varepsilon_\perp \longrightarrow \mathbb{P}_\perp v_\perp = v_\perp \quad \text{in} \quad L^2_{\text{loc}}(\mathbb{R}_+; L^p \cap H^s(\mathbb{T}^3))-\text{strong}\,, \quad 1 \leq p < 6\,, \quad 0 \leq s < 1\,,\\
& \rho^\varepsilon v^\varepsilon \rightharpoonup v \quad \text{in} \quad L^2_{\text{loc}}(\mathbb{R}_+; L^q \cap H^{-\sigma}(\mathbb{T}^3))-\text{weak}\,, \quad \forall\, \sigma \geq \mathfrak{s} := \max\Big\{\frac{1}{2}, \frac{3}{\gamma} - 1\Big\}\,, \quad q = \frac{6\gamma}{6+\gamma}\,,\\
& \rho^\varepsilon v^\varepsilon - v^\varepsilon \longrightarrow 0 \quad \text{in} \quad L^2_{\text{loc}}(\mathbb{R}_+; L^q(\mathbb{T}^3))-\text{strong}\,, \quad q = 6\gamma/(6+\gamma)\,,\\
& \mathbb{P}_\perp(\rho^\varepsilon v^\varepsilon_\perp) \longrightarrow \mathbb{P}_\perp v_\perp = v_\perp \quad \text{in} \quad L^2_{\text{loc}}(\mathbb{R}_+; L^q(\mathbb{T}^3))-\text{strong}\,, \quad q = 6\gamma/(6+\gamma)\,.
\end{aligned}
$$

Proof. We start with the first statement of Lemma 3. From energy inequality (30)-(33) with the pressure term Π_2, we obtain that v^ε is bounded in $L^2_{\text{loc}}(\mathbb{R}_+; \dot{H}^1(\mathbb{T}^3))$. Let us show that v^ε is bounded in $L^2_{\text{loc}}(\mathbb{R}_+; H^1(\mathbb{T}^3))$. From Poincaré–Wirtinger inequality and Hölder inequality, it is easy to show that

$$
\|\cdot\|^2_{H^1(\mathbb{T}^3)} \sim |\overline{(\cdot)}|^2 + \|\cdot\|^2_{\dot{H}^1(\mathbb{T}^3)}\,, \qquad \text{with} \quad \overline{(\cdot)} \equiv |\mathbb{T}^3|^{-1} \int_{\mathbb{T}^3} dx\, \cdot\,.
$$

There remains to control the mean value $\overline{v^\varepsilon}$. Using Hölder inequality, the embedding $L^6(\mathbb{T}^3) \hookrightarrow L^{2\gamma/(\gamma-1)}(\mathbb{T}^3)$ for $\gamma > 3/2$, the continuous embedding $H^1_0(\mathbb{T}^3) \hookrightarrow L^6(\mathbb{T}^3)$ (with $H^1_0(\mathbb{T}^3)$ the set of zero-average functions in $\dot{H}^1(\mathbb{T}^3)$), we obtain for any $T > 0$,

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$$\begin{aligned}\int_0^T dt \int_{\mathbb{T}^3} dx\, \rho^\varepsilon \big|v^\varepsilon - \overline{v^\varepsilon}\big|^2 &\leq \int_0^T dt\, \|\rho^\varepsilon\|_{L^\gamma(\mathbb{T}^3)} \big\|v^\varepsilon - \overline{v^\varepsilon}\big\|^2_{L^{2\gamma/(\gamma-1)}(\mathbb{T}^3)} \\ &\leq \int_0^T dt\, \|\rho^\varepsilon\|_{L^\gamma(\mathbb{T}^3)} \big\|v^\varepsilon - \overline{v^\varepsilon}\big\|^2_{L^6(\mathbb{T}^3)} \\ &\leq \|\rho^\varepsilon\|_{L^\infty_{\mathrm{loc}}(\mathbb{R}_+;L^\gamma(\mathbb{T}^3))} \|\nabla v^\varepsilon\|^2_{L^2_{\mathrm{loc}}(\mathbb{R}_+;L^2(\mathbb{T}^3))} \leq C < \infty\,. \end{aligned} \tag{43}$$

Using inequality $b^2/2 \leq (a-b)^2 + a^2$ and the mass conservation law, we obtain

$$\int_{\mathbb{T}^3} dx\, \rho^\varepsilon (\overline{v^\varepsilon})^2 = \|\rho_0^\varepsilon\|_{L^1(\mathbb{T}^3)} (\overline{v^\varepsilon})^2 \leq 2 \int_{\mathbb{T}^3} dx\, \rho^\varepsilon (v^\varepsilon - \overline{v^\varepsilon})^2 + 2 \int_{\mathbb{T}^3} dx\, \rho^\varepsilon (v^\varepsilon)^2.$$

With (43), this implies (uniformly in ε)

$$(\overline{v^\varepsilon})^2 \leq 2\, T^{-1} \, \|\rho_0^\varepsilon\|^{-1}_{L^1(\mathbb{T}^3))} \big(C + T \, \|\sqrt{\rho^\varepsilon} v^\varepsilon\|^2_{L^\infty([0,T];L^2(\mathbb{T}^3))} \big) \leq \widetilde{C}\,.$$

This information combined with the bound of v^ε in $L^2_{\mathrm{loc}}(\mathbb{R}_+;\dot{H}^1(\mathbb{T}^3))$ indicates that v^ε is bounded in $L^2_{\mathrm{loc}}(\mathbb{R}_+;H^1(\mathbb{T}^3))$.

We continue with the second statement. To this end, we look at the mass conservation law (20). Since $\rho^\varepsilon \to 1$ in $L^\infty_{\mathrm{loc}}(\mathbb{R}_+;L^\gamma(\mathbb{T}^3))$−strong (Lemma 1), $v^\varepsilon \rightharpoonup v$ in $L^2_{\mathrm{loc}}(\mathbb{R}_+;L^6(\mathbb{T}^3))$−weak and $\rho_0^\varepsilon \to 1$ in $L^\gamma(\mathbb{T}^3)$−strong (from (27) and by following the proof of Lemma 1), it is easy to pass to the limit in the distributional sense in the linear terms. For the nonlinear term, we write $\rho^\varepsilon\, v^\varepsilon = (\rho^\varepsilon - 1)\, v^\varepsilon + v^\varepsilon$. Since $1/\gamma + 1/6 < 1$ (recall that $\gamma > 3/2$), the first term vanishes strongly in $L^2(\mathbb{R}_+;L^{6\gamma/(6+\gamma)}(\mathbb{T}^3))$. At the limit, we recover for any test function φ that

$$\int_\Omega dx\, \varphi(0) + \int_0^\infty dt \int_\Omega dx\, \big(\partial_t \varphi + v_\perp \cdot \nabla_\perp \varphi\big) = \int_0^\infty dt \int_\Omega dx\, v_\perp \cdot \nabla_\perp \varphi = 0\,,$$

which means that $\nabla_\perp \cdot v_\perp = 0$ in $\mathcal{D}'(\mathbb{R}_+^* \times \mathbb{T}^3)$.

The third assertion of Lemma 3 is a consequence of the first and second statements of Lemma 3, of the Helmhotz–Hodge decomposition $v_\perp^\varepsilon = \mathbb{P}_\perp v_\perp^\varepsilon + \mathbb{Q}_\perp v_\perp^\varepsilon$ and of the (weak) continuity properties of $\mathbb{P}_\perp$ and $\mathbb{Q}_\perp$.

The fourth assertion exploits some Gagliardo–Nirenberg interpolation inequalities together with delicate equicontinuity properties in time that require to already control the product $\rho^\varepsilon\, v^\varepsilon$. The proof is postponed to a later stage.

We pursue with the proof of fifth statement of Lemma 3. On the one hand, from the uniform bounds $v^\varepsilon \in L^2_{\mathrm{loc}}(\mathbb{R}_+;L^6(\mathbb{T}^3))$ and $\rho^\varepsilon \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+;L^\gamma(\mathbb{T}^3))$, and on the other hand, from Hölder inequality, we obtain $\rho^\varepsilon v^\varepsilon \in L^2_{\mathrm{loc}}(\mathbb{R}_+;L^{6\gamma/(6+\gamma)}(\mathbb{T}^3))$ uniformly with respect to ε, which gives, by weak compactness, the weak convergence of this sequence in this space. Moreover, from the first assertion of Lemma 1 and 3, the product of ρ^ε and v^ε weakly converges to the limit point v in $L^2_{\mathrm{loc}}(\mathbb{R}_+;L^q(\mathbb{T}^3))$−weak with $1/q = 1/\gamma + 1/6$. The Sobolev embedding $H^{\mathfrak{s}}(\mathbb{T}^3) \hookrightarrow L^{q'}(\mathbb{T}^3)$, with $1/q' = 1 - 1/q = (5\gamma - 6)/(6\gamma)$ and $\mathfrak{s} \geq \max\{0, 3/\gamma - 1\}$, implies by duality that $L^q(\mathbb{T}^3) \hookrightarrow H^{-\mathfrak{s}}(\mathbb{T}^3)$. Without loss of generality and in order to avoid further the multiplication of regularity indices we restrict $\mathfrak{s}$ such that $\mathfrak{s} \geq \max\{1/2, 3/\gamma - 1\}$. We then have $\rho^\varepsilon v^\varepsilon \in L^2_{\mathrm{loc}}(\mathbb{R}_+;H^{-\mathfrak{s}}(\mathbb{T}^3))$. We conclude by using the Sobolev embedding $H^\sigma(\mathbb{T}^3) \hookrightarrow H^{\mathfrak{s}}(\mathbb{T}^3)$ for $\sigma \geq \mathfrak{s}$, and duality.

From Hölder inequality, we have

$$\|\rho^\varepsilon v^\varepsilon - v^\varepsilon\|_{L^2_{\mathrm{loc}}(\mathbb{R}_+;L^q(\mathbb{T}^3))} \leq \|\rho^\varepsilon - 1\|_{L^\infty_{\mathrm{loc}}(\mathbb{R}_+;L^\gamma(\mathbb{T}^3))} \|v^\varepsilon\|_{L^2_{\mathrm{loc}}(\mathbb{R}_+;L^6(\mathbb{T}^3))}\,, \qquad 1/q = 1/\gamma + 1/6\,.$$

Then, exploiting the first assertion of Lemmas 1 and 3, we obtain the sixth statement of Lemma 3.

We can now come back to the $L^2_{\text{loc}}(\mathbb{R}_+; L^p(\mathbb{T}^3))-$strong convergence in the fourth assertion of Lemma 3. For this, we use Lemma 14 of Appendix A, for which we show below that its hypotheses, splitted in three points, are satisfied. 1) From the fifth statement of Lemma 3, the selfadjointness of $\mathbb{P}_\perp$ for the scalar product of $L^2(\Omega; \mathbb{R}^2)$, and the continuity of $\mathbb{P}_\perp$ in $L^\alpha(\Omega; \mathbb{R}^2)$, for $1 < \alpha < \infty$, we obtain $\mathbb{P}_\perp(\rho^\varepsilon v^\varepsilon_\perp) \rightharpoonup \mathbb{P}_\perp v_\perp = v_\perp$ in $L^2_{\text{loc}}(\mathbb{R}_+; L^{6\gamma/(6+\gamma)}(\mathbb{T}^3))-$weak. 2) The bound $\mathbb{P}_\perp v^\varepsilon_\perp \in L^2_{\text{loc}}(\mathbb{R}_+; L^6(\mathbb{T}^3))$ and Lemma 4.3 in [7] implies $\|\mathbb{P}_\perp v^\varepsilon_\perp(t, \cdot + h) - \mathbb{P}_\perp v^\varepsilon_\perp(t, \cdot)\|_{L^2_{\text{loc}}(\mathbb{R}_+; L^6(\mathbb{T}^3))} \to 0$, as $|h| \to 0$, uniformly with respect to ε. 3) Applying the Leray projector $\mathbb{P}_\perp$ to equation (21) for $\rho^\varepsilon v^\varepsilon_\perp$, we obtain

$$\partial_t(\mathbb{P}_\perp(\rho^\varepsilon v^\varepsilon_\perp)) = -\nabla_\varepsilon \cdot \mathbb{P}_\perp(\rho^\varepsilon v^\varepsilon_\perp \otimes v^\varepsilon) + \partial_\parallel \mathbb{P}_\perp B^\varepsilon - \nabla_\varepsilon \cdot \mathbb{P}_\perp(B^\varepsilon_\perp \otimes B^\varepsilon) + \mu^\varepsilon_\perp \Delta_\perp \mathbb{P}_\perp v^\varepsilon_\perp + \mu^\varepsilon_\parallel \Delta_\parallel \mathbb{P}_\perp v^\varepsilon_\perp \,. \quad (44)$$

Using the bounds $\rho^\varepsilon |v^\varepsilon|^2 \in L^\infty_{\text{loc}}(\mathbb{R}_+; L^1(\mathbb{T}^3))$, $v^\varepsilon \in L^2_{\text{loc}}(\mathbb{R}_+; H^1(\mathbb{T}^3))$, and $B^\varepsilon \in L^\infty_{\text{loc}}(\mathbb{R}_+; L^2(\mathbb{T}^3))$, and the following properties of the projector $\mathbb{P}_\perp$, $\mathbb{P}_\perp(H^\alpha(\mathbb{T}^3)) \hookrightarrow H^\alpha(\mathbb{T}^3)$, with $\alpha \geq 0$, and $\mathbb{P}_\perp(L^1(\mathbb{T}^3)) \hookrightarrow W^{-\delta,1}(\mathbb{T}^3)$, with $\delta > 0$, we obtain from (44), $\partial_t(\mathbb{P}_\perp(\rho^\varepsilon v^\varepsilon_\perp)) \in L^2_{\text{loc}}(\mathbb{R}_+; (W^{-\delta-1,1} + H^{-1} + L^2)(\mathbb{T}^3)) \hookrightarrow L^1_{\text{loc}}(\mathbb{R}_+; W^{-\delta-1,1}(\mathbb{T}^3))$. Gathering points 1) to 3), we can apply Lemma 14 of Appendix A with $g^\varepsilon = \rho^\varepsilon v^\varepsilon_\perp$, $h^\varepsilon = \mathbb{P}_\perp v^\varepsilon_\perp$, $p_1 = q_1 = 2$ $(1/p_1 + 1/q_1 = 1)$, $p_2 = 6\gamma/(6+\gamma)$, and $q_2 = 6$ $(1/p_2 + 1/q_2 = 1/\gamma + 1/3 < 1$, for $\gamma > 3/2)$, to deduce that

$$\mathbb{P}_\perp(\rho^\varepsilon v^\varepsilon_\perp) \cdot \mathbb{P}_\perp v^\varepsilon_\perp \longrightarrow |\mathbb{P}_\perp v_\perp|^2 = |v_\perp|^2 \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^*_+ \times \mathbb{T}^3) \,. \quad (45)$$

This limit leads to the strong convergence of $\mathbb{P}_\perp v^\varepsilon_\perp$ to $v_\perp$ in $L^2_{\text{loc}}(\mathbb{R}_+; L^2(\mathbb{T}^3))$. Indeed, using (45) and Hölder inequality, we obtain for any $T > 0$,

$$\begin{aligned} &\limsup_{\varepsilon \to 0} \|\mathbb{P}_\perp v^\varepsilon_\perp\|^2_{L^2_{\text{loc}}(\mathbb{R}_+; L^2(\mathbb{T}^3))} - \|\mathbb{P}_\perp v_\perp\|^2_{L^2_{\text{loc}}(\mathbb{R}_+; L^2(\mathbb{T}^3))} \\ &\quad = \limsup_{\varepsilon \to 0} \int_0^T dt \int_{\mathbb{T}^3} dx \left(|\mathbb{P}_\perp v^\varepsilon_\perp|^2 - \mathbb{P}_\perp(\rho^\varepsilon v^\varepsilon_\perp) \cdot \mathbb{P}_\perp v^\varepsilon_\perp \right) = \limsup_{\varepsilon \to 0} \int_0^T dt \int_{\mathbb{T}^3} dx \, \mathbb{P}_\perp v^\varepsilon_\perp \cdot v^\varepsilon_\perp (1 - \rho^\varepsilon) \\ &\quad \leq \limsup_{\varepsilon \to 0} \|\rho^\varepsilon - 1\|_{L^\infty_{\text{loc}}(\mathbb{R}_+; L^\gamma(\mathbb{T}^3))} \|v^\varepsilon_\perp\|^2_{L^2_{\text{loc}}(\mathbb{R}_+; L^{2\theta}(\mathbb{T}^3))} \,, \end{aligned}$$

with $1/\gamma + 1/\theta = 1$. Since $\gamma > 3/2$, we obtain $2\theta < 6$. Therefore, using the first statement of Lemmas 1 and 3, the right-hand side of the previous inequality vanishes, which leads to $\limsup_{\varepsilon \to 0} \|\mathbb{P}_\perp v^\varepsilon_\perp\|_{L^2_{\text{loc}}(\mathbb{R}_+; L^2(\mathbb{T}^3))} \leq \|\mathbb{P}_\perp v_\perp\|_{L^2_{\text{loc}}(\mathbb{R}_+; L^2(\mathbb{T}^3))} \geq \liminf_{\varepsilon \to 0} \|\mathbb{P}_\perp v^\varepsilon_\perp\|_{L^2_{\text{loc}}(\mathbb{R}_+; L^2(\mathbb{T}^3))}$ and to $\mathbb{P}_\perp v^\varepsilon_\perp \to \mathbb{P}_\perp v_\perp = v_\perp$ in $L^2_{\text{loc}}(\mathbb{R}_+; L^2(\mathbb{T}^3))-$strong, by Proposition 3.32 in [7]. This strong convergence in L^2 allows to prove the fourth statement of Lemma 3 by using some interpolation inequalities. Indeed, using Gagliardo–Nirenberg inequality in space and Cauchy-Schwarz inequality in time, we obtain

$$\|\mathbb{P}_\perp v^\varepsilon_\perp - v_\perp\|_{L^2_{\text{loc}}(\mathbb{R}_+; L^{p_1}(\mathbb{T}^3))} \lesssim \|\mathbb{P}_\perp v^\varepsilon_\perp - v_\perp\|^{1/2}_{L^2_{\text{loc}}(\mathbb{R}_+; \dot{H}^1(\mathbb{T}^3))} \|\mathbb{P}_\perp v^\varepsilon_\perp - v_\perp\|^{1/2}_{L^2_{\text{loc}}(\mathbb{R}_+; L^{p_0}(\mathbb{T}^3))} \,, \quad (46)$$

with $1/p_1 = 1/12 + 1/(2p_0)$ and $\|\mathbb{P}_\perp v^\varepsilon_\perp - v_\perp\|_{L^2_{\text{loc}}(\mathbb{R}_+; \dot{H}^1(\mathbb{T}^3))} < \infty$. Iterating inequality (46), we obtain a sequence of indices p_n such that $1/p_{n+1} = 1/12 + 1/(2p_n)$, with $p_0 = 2$, hence its limit $p_\infty = 6$. This justifies the first part (strong convergence in L^p) of the fourth statement of Lemma 3. For the second part of this statement, again, using Gagliardo–Nirenberg inequality in space (with not necessarily integers [8,9]) and Cauchy-Schwarz inequality in time, we obtain

$$\|\mathbb{P}_\perp v^\varepsilon_\perp - v_\perp\|_{L^2_{\text{loc}}(\mathbb{R}_+; H^{s_1}(\mathbb{T}^3))} \lesssim \|\mathbb{P}_\perp v^\varepsilon_\perp - v_\perp\|^{1/2}_{L^2_{\text{loc}}(\mathbb{R}_+; H^1(\mathbb{T}^3))} \|\mathbb{P}_\perp v^\varepsilon_\perp - v_\perp\|^{1/2}_{L^2_{\text{loc}}(\mathbb{R}_+; H^{s_0}(\mathbb{T}^3))} \,, \quad (47)$$

with $s_1 = 1/2 + s_0/2$ and $\|\mathbb{P}_\perp v^\varepsilon_\perp - v_\perp\|_{L^2_{\text{loc}}(\mathbb{R}_+; H^1(\mathbb{T}^3))} < \infty$. Iterating inequality (47), we obtain a sequence of indices s_n such that $s_{n+1} = 1/2 + s_n/2$, with $s_0 = 0$, hence its limit $s_\infty = 1$. This justifies the second part (strong convergence in H^s) of the fourth statement of Lemma 3.

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We finish with the proof the seventh statement. By triangular inequality and continuity of $\mathbb{P}_\perp$ in $L^\alpha(\Omega;\mathbb{R}^2)$, for $1<\alpha<\infty$, we obtain

$$\|\mathbb{P}_\perp(\rho^\varepsilon v_\perp^\varepsilon - v_\perp)\|_{L^2_{\mathrm{loc}}(\mathbb{R}_+;L^q(\mathbb{T}^3))} \leq \|\mathbb{P}_\perp v_\perp^\varepsilon - v_\perp\|_{L^2_{\mathrm{loc}}(\mathbb{R}_+;L^q(\mathbb{T}^3))} + \|\rho^\varepsilon v_\perp^\varepsilon - v_\perp^\varepsilon\|_{L^2_{\mathrm{loc}}(\mathbb{R}_+;L^q(\mathbb{T}^3))}.$$

Using the fourth and sixth assertions of Lemma 3, and since $6/5<q<6$ (because $1/q=1/\gamma+1/6$, and $3/2<\gamma<\infty$) this inequality allows to conclude. □

4.4. Passage to the limit in the equation for $B_\perp^\varepsilon$

Here, we justify the passage to the limit in (23) for $B_\perp^\varepsilon$. Let us start with the initial condition term. From the discussion about the properties of sequences of initial conditions in Section 2.3 (in particular the uniform bound (27) and the resulting convergences), we can pass to the limit, in the distributional sense, in the initial condition term of (23) to obtain the limit initial condition $B_{0\perp}$. Next, using on the one hand the third statement of Lemma 2, and on the other hand the first and the second statements of Lemma 3, we can pass to the limit, in the distributional sense, in all linear and nonlinear terms of equations (23) and (25) to obtain the first equation of (12) and equation (15) in the sense of distributions.

4.5. Passage to the limit in the equation for $\rho^\varepsilon v_\perp^\varepsilon$

Here, we justify the passage to the limit in equation (21) for $\rho^\varepsilon v_\perp^\varepsilon$, in several steps. We start by recalling some basic tools. With Ω being either $\mathbb{T}^3$ or $\mathbb{R}^3$, we introduce the linear differential operator

$$\mathcal{L} := -\begin{pmatrix} 0 & c^\varepsilon\,\nabla_\perp\cdot \\ {}^t\nabla_\perp & 0 \end{pmatrix}, \tag{48}$$

defined on $\mathcal{D}'_0(\Omega;\mathbb{R})\times\mathcal{D}'(\Omega;\mathbb{R}^2)$, where $\mathcal{D}'_0(\Omega;\mathbb{R})=\{\phi\in\mathcal{D}'(\Omega;\mathbb{R})\ |\ \int_\Omega dx\,\phi=0\}$, and such that $\mathcal{L}U=-\,{}^t(c^\varepsilon\nabla_\perp\cdot\Phi,\ {}^t\nabla_\perp\phi)$, with $U\equiv{}^t(\phi,{}^t\Phi)\in\mathcal{D}'_0(\Omega;\mathbb{R})\times\mathcal{D}'(\Omega;\mathbb{R}^2)$. Here $c^\varepsilon:=b^\varepsilon+1/\overline{\rho^\varepsilon}$, with $b^\varepsilon:=b(\overline{\rho^\varepsilon})^{\gamma-1}$. Since for ε small enough $\overline{\rho^\varepsilon}\in(1/2,3/2)$, there exist constants $0<\underline{c}\leq\bar{c}<\infty$, such that $\underline{c}\leq c^\varepsilon\leq\bar{c}$, and $c^\varepsilon\to c=1+b=1+a\gamma$.

We claim that $\mathcal{L}$ generates a one-parameter group of isometry $\{\mathcal{S}(\tau):=\exp(\tau\mathcal{L});\ \tau\in\mathbb{R}\}$, from $H^\alpha(\Omega;\mathbb{R})\times H^\alpha(\Omega;\mathbb{R}^2)$ into itself with the norm $|||U|||^2_{H^\alpha(\Omega)}:=\|\phi\|^2_{H^\alpha(\Omega)}+c^\varepsilon\|\Phi\|^2_{H^\alpha(\Omega)}$, for all $\alpha\in\mathbb{R}$. Indeed, this comes from the fact that the operator $\mathcal{L}$ is skew-adjoint for the scalar product $\langle\!\langle\cdot\,,\,\cdot\rangle\!\rangle$ of $L^2(\Omega;\mathbb{R})\times L^2(\Omega;\mathbb{R}^2)$, defined by $\langle\!\langle U,\,V\rangle\!\rangle=\langle\phi,\psi\rangle_{L^2(\Omega)}+c^\varepsilon\langle\Phi,\Psi\rangle_{L^2(\Omega)}$, where $U\equiv{}^t(\phi,{}^t\Phi)$, $V\equiv{}^t(\psi,{}^t\Psi)$, and the notation $\langle\cdot,\cdot\rangle_{L^2(\Omega)}$ stands for the standard scalar product in $L^2(\Omega)$ for scalar or vector valued functions. This isometry group can also be directly verified from the H^α-energy estimates of the solutions $U(\tau)=\mathcal{S}(\tau)\,U_0$, satisfying the equation $\partial_\tau U(\tau)=\mathcal{L}\,U(\tau)$, i.e.,

$$\partial_\tau\phi+c^\varepsilon\,\nabla_\perp\cdot\Phi=0\,,\quad \partial_\tau\Phi+\nabla_\perp\phi=0\,. \tag{49}$$

Indeed, we first set $\Lambda=(\mathrm{I}-\Delta)^{1/2}$ and recall that $H^\alpha(\Omega)=\Lambda^{-\alpha}L^2(\Omega)$. Applying Λ^α to equations (49), and taking the $L^2(\Omega)$ scalar product of the result with $\Lambda^\alpha\phi$ (resp. $c^\varepsilon\Lambda^\alpha\Phi$) for the first (resp. second) equation of (49), we obtain

$$\begin{aligned}\frac{d}{dt}|||U|||^2_{H^\alpha(\Omega)} &= \frac{d}{dt}\langle\!\langle\Lambda^\alpha U,\Lambda^\alpha U\rangle\!\rangle=\frac{d}{dt}\big(\|\Lambda^\alpha\phi\|^2_{L^2(\Omega)}+c^\varepsilon\|\Lambda^\alpha\Phi\|^2_{L^2(\Omega)}\big)\\ &= \int_\Omega dx\,c^\varepsilon\big(\Lambda^\alpha\phi\nabla_\perp\cdot\Lambda^\alpha\Phi+\Lambda^\alpha\Phi\cdot\nabla_\perp\Lambda^\alpha\phi\big)=0\,.\end{aligned}$$

In order to understand some properties of the group $\mathcal{S}(\tau)$, we denote by $\mathcal{S}_1(\tau) \in \mathbb{R}$ and $\mathcal{S}_2(\tau) \in \mathbb{R}^2$ the components of $\mathcal{S}(\tau)$. Observe that $\int_\Omega dx\, \mathcal{S}_1(\tau)\, U$ and $\mathbb{P}_\perp \mathcal{S}_2(\tau)\, U$ are independent of $\tau \in \mathbb{R}$. In particular $\mathcal{S}(\tau)\, U$ is independent of τ, if $\mathbb{P}_\perp \Phi = 0$ (i.e., $\nabla_\perp \cdot \Phi = 0$) and if ϕ is constant. From (49) the operator $\mathcal{L}$ is equivalent to the transverse wave operator. Indeed, the equation $\partial_\tau U = \mathcal{L}\, U$ is equivalent to the scalar wave equations $(\partial_\tau^2 - c^\varepsilon \Delta_\perp)\phi = 0$, and $(\partial_\tau^2 - c^\varepsilon \Delta_\perp)\varphi = 0$, where we have used the Helmholtz–Hodge decomposition $\Phi = \mathbb{P}_\perp \Phi + \nabla_\perp \varphi$, with $\int_\Omega dx\, \varphi = 0$, $\forall\, \tau \in \mathbb{R}$, and observing that $\mathbb{P}_\perp \Phi$ is a constant determined by the initial conditions (since $\partial_\tau \mathbb{P}_\perp \Phi = 0$ from the second equation of (49)).

The key to justify the limit of the equation for $\rho^\varepsilon v_\perp^\varepsilon$, is to construct an approximate solution to the MHD equations (20)-(25), which allows us to pass to the limit in both singular terms and nonlinear terms. Such a construction is given by the following lemma.

Lemma 4. *Let us define $U^\varepsilon := {}^t(\phi^\varepsilon, {}^t\Phi^\varepsilon)$, where $\phi^\varepsilon := b^\varepsilon \varrho^\varepsilon + B_\parallel^\varepsilon$, and $\Phi^\varepsilon := \mathbb{Q}_\perp(\rho^\varepsilon v_\perp^\varepsilon)$, with $\varrho^\varepsilon := (\rho^\varepsilon - \overline{\rho^\varepsilon})/\varepsilon$, and $b^\varepsilon := b(\overline{\rho^\varepsilon})^{\gamma-1}$. Let $\mathfrak{s}$ be the same positive real number as in Lemma 3, i.e., $\mathfrak{s} := \max\{1/2, 3/\gamma - 1\} \in [1/2, 1)$. Then,*

1. *There exist functions $\mathcal{U} = {}^t(\psi, {}^t\Psi) \in L^2_{\mathrm{loc}}(\mathbb{R}_+; L^2(\mathbb{T}^3; \mathbb{R}^3))$, and $\mathcal{R}^\varepsilon \in L^2_{\mathrm{loc}}(\mathbb{R}_+; H^{-\sigma}(\mathbb{T}^3, \mathbb{R}^3))$, with $\mathfrak{s} < \sigma < (5/2)^+$, such that*
$$U^\varepsilon = \mathcal{S}(t/\varepsilon)\mathcal{U} + \mathcal{R}^\varepsilon, \quad \textit{with} \;\; \mathcal{R}^\varepsilon \longrightarrow 0 \quad \text{in} \;\; L^2_{\mathrm{loc}}(\mathbb{R}_+; H^{-\sigma}(\mathbb{T}^3))-\text{strong}\,. \tag{50}$$
2. *There exists a function $\pi_0 := \Delta_\perp^{-1} \nabla_\perp \cdot \mathbb{Q}_\perp u_{0\perp} \in L^2(\mathbb{T}_\parallel; H^1(\mathbb{T}_\perp^2))$, as well as a function $\pi_1 \in L^2_{\mathrm{loc}}(\mathbb{R}_+; H^{-r}(\mathbb{T}^3; \mathbb{R}))$, with $r > \left(\frac{5}{2}\right)^+$, such that*
$$\phi^\varepsilon/\varepsilon \longrightarrow \delta_0(t) \otimes \pi_0 \, + \, \pi_1 \quad \text{in} \;\; H^{-1}(\mathbb{R}_+; H^{-r}(\mathbb{T}^3))-\text{weak}\,. \tag{51}$$
3. *The limit point $(B_\parallel, \varrho)$ satisfies the relation $B_\parallel + b\varrho = 0$, for a.e. $(t, x) \in]0, +\infty[\times \mathbb{T}^3$.*

Proof. On the one hand, applying the projector $\mathbb{Q}_\perp$ to the perpendicular component of the second equation of (3) (the one for $\rho^\varepsilon v_\perp^\varepsilon$) to form an equation for $\Phi^\varepsilon := \mathbb{Q}_\perp(\rho^\varepsilon v_\perp^\varepsilon)$ and on the other hand combining the first equation of (3) (the one for ρ^ε) and the parallel component of the second equation of (3) (the one for $B_\parallel^\varepsilon$) to form an equation for $\phi^\varepsilon := b^\varepsilon \varrho^\varepsilon + B_\parallel^\varepsilon$, we obtain the following equation for $U^\varepsilon := {}^t(\phi^\varepsilon, {}^t\Phi^\varepsilon)$,
$$\partial_t U^\varepsilon - \frac{1}{\varepsilon}\mathcal{L} U^\varepsilon = F^\varepsilon\,, \tag{52}$$
where
$$F^\varepsilon := \begin{pmatrix} F_1^\varepsilon \\ F_2^\varepsilon \end{pmatrix} = \begin{cases} F_1^\varepsilon = -\, b^\varepsilon \partial_\parallel(\rho^\varepsilon v_\parallel^\varepsilon) + (\overline{\rho^\varepsilon})^{-1} \nabla_\perp \cdot \mathbb{Q}_\perp(\varrho^\varepsilon v_\perp^\varepsilon) - B_\parallel^\varepsilon \nabla_\varepsilon \cdot v^\varepsilon \\ \qquad -\, (v^\varepsilon \cdot \nabla_\varepsilon) B_\parallel^\varepsilon + (B^\varepsilon \cdot \nabla_\varepsilon) v_\parallel^\varepsilon + (\eta_\perp^\varepsilon \Delta_\perp + \eta_\parallel^\varepsilon \Delta_\parallel) B_\parallel^\varepsilon\,, \\ F_2^\varepsilon = -\, \mathbb{Q}_\perp \nabla_\varepsilon \cdot (\rho^\varepsilon v_\perp^\varepsilon \otimes v^\varepsilon) - (\gamma - 1)\nabla_\perp \Pi_2(\rho^\varepsilon) - \frac{1}{2}\nabla_\perp(|B^\varepsilon|^2) + \partial_\parallel \mathbb{Q}_\perp B_\perp^\varepsilon \\ \qquad +\, \mathbb{Q}_\perp \nabla_\varepsilon \cdot (B_\perp^\varepsilon \otimes B^\varepsilon) + \mu_\perp^\varepsilon \nabla_\perp(\nabla_\perp \cdot v_\perp^\varepsilon) + \mu_\parallel^\varepsilon \Delta_\parallel \mathbb{Q}_\perp v_\perp^\varepsilon + \lambda^\varepsilon \nabla_\perp(\nabla_\varepsilon \cdot v^\varepsilon)\,. \end{cases} \tag{53}$$

We start with point 1 of Lemma 4. We aim at showing (50) in two steps. The first step concerns the existence of a filtered profile $\mathcal{U} = {}^t(\psi, {}^t\Psi)$, while the second one establishes its regularity in L^2.

Step 1. Here, we show that the filtered solution ${}^t(\psi^\varepsilon, {}^t\Psi^\varepsilon) \equiv \mathcal{U}^\varepsilon := \mathcal{S}(-t/\varepsilon)U^\varepsilon$ is relatively compact in $L^2_{\mathrm{loc}}(\mathbb{R}_+; H^{-\sigma}(\mathbb{T}^3))$ for $\sigma > \mathfrak{s}$. For this, we first show uniform bounds for U^ε in suitable functional spaces. Second, we use the fact that the group $\mathcal{S}$ is an isometry in H^α ($\alpha \in \mathbb{R}$) in order to obtain similar bounds on $(\mathcal{U}^\varepsilon, \partial_t \mathcal{U}^\varepsilon)$. Next, we invoke an Aubin–Lions theorem to obtain compactness of the sequence $\mathcal{U}^\varepsilon$ and the existence of the filtered profile $\mathcal{U}$. Finally, using the isometry $\mathcal{S}$ and the averaged profile $\mathcal{U}$, we construct an approximation to U^ε, with an error estimate which converges strongly as indicated in (50). In this way, we bypass the singularity in $1/\varepsilon$ in (52).

Recall that $\kappa = \min\{2, \gamma\}$. Since $\varrho^\varepsilon \in L^\infty_{\text{loc}}(\mathbb{R}_+; L^\kappa(\mathbb{T}^3))$ and $B^\varepsilon_\parallel \in L^\infty_{\text{loc}}(\mathbb{R}_+; L^2(\mathbb{T}^3))$, we can assert that $\phi^\varepsilon \in L^\infty_{\text{loc}}(\mathbb{R}_+; (L^\kappa + L^2)(\mathbb{T}^3))$. From Sobolev embeddings and a duality argument, we obtain $L^{3/2}(\mathbb{T}^3) \hookrightarrow H^{-\alpha}(\mathbb{T}^3)$, with $\alpha \geq 1/2$. Since $\kappa > 3/2$, then $L^\kappa(\mathbb{T}^3) \hookrightarrow H^{-\alpha}(\mathbb{T}^3)$ and $(L^\kappa + L^2)(\mathbb{T}^3) \hookrightarrow H^{-\alpha}(\mathbb{T}^3)$. Since α can be chosen such that $\alpha \leq \mathfrak{s}$, then $(L^\kappa + L^2)(\mathbb{T}^3) \hookrightarrow H^{-\mathfrak{s}}(\mathbb{T}^3)$ and $\phi^\varepsilon \in L^\infty_{\text{loc}}(\mathbb{R}_+; H^{-\mathfrak{s}}(\mathbb{T}^3)) \hookrightarrow L^2_{\text{loc}}(\mathbb{R}_+; H^{-\mathfrak{s}}(\mathbb{T}^3))$. From the fifth statement of Lemma 3, $\rho^\varepsilon v^\varepsilon_\perp \in L^2_{\text{loc}}(\mathbb{R}_+; H^{-\mathfrak{s}}(\mathbb{T}^3))$. Since $\mathbb{Q}_\perp$ (resp. $\mathcal{S}$) is a continuous map (resp. an isometry) in H^α, with $\alpha \in \mathbb{R}$, then $\mathcal{U}^\varepsilon \in L^2_{\text{loc}}(\mathbb{R}_+; H^{-\mathfrak{s}}(\mathbb{T}^3))$.

We are now going to obtain a bound for $\partial_t \mathcal{U}^\varepsilon$. Using (52), a straightforward computation shows that $\partial_t \mathcal{U}^\varepsilon = \mathcal{S}(-t/\varepsilon) F^\varepsilon$. Let us show that $F^\varepsilon \in L^2_{\text{loc}}(\mathbb{R}_+; H^{-r}(\mathbb{T}^3))$ for $r > 5/2 + \delta$, and any $\delta > 0$. We will then obtain $\partial_t \mathcal{U}^\varepsilon \in L^2_{\text{loc}}(\mathbb{R}_+; H^{-r}(\mathbb{T}^3))$, because $\mathcal{S}$ is an unitary group in H^α, with $\alpha \in \mathbb{R}$.

Let us start with F^ε_1. First, observe that $B^\varepsilon_\parallel \nabla_\varepsilon \cdot v^\varepsilon + (v^\varepsilon \cdot \nabla_\varepsilon) B^\varepsilon_\parallel - (B^\varepsilon \cdot \nabla_\varepsilon) v^\varepsilon_\parallel = [\nabla_\varepsilon \times (B^\varepsilon_\parallel \times v^\varepsilon)]_\parallel \in L^2_{\text{loc}}(\mathbb{R}_+; W^{-1,3/2}(\mathbb{T}^3))$ by using Hölder inequality and the energy estimate. Obviously, from the energy estimate, the last two diffusive terms of F^ε_1 belong to $L^2_{\text{loc}}(\mathbb{R}_+; H^{-1}(\mathbb{T}^3))$. Using $U^\varepsilon \in L^2_{\text{loc}}(\mathbb{R}_+; H^{-\mathfrak{s}}(\mathbb{T}^3))$, the first term of F^ε_1 is in $L^2_{\text{loc}}(\mathbb{R}_+; H^{-\mathfrak{s}-1}(\mathbb{T}^3))$. It remains to bound the second term of F^ε_1. Using Hölder inequality, we obtain $\varrho^\varepsilon v^\varepsilon \in L^2_{\text{loc}}(\mathbb{R}_+; L^{\mathfrak{q}}(\mathbb{T}^3))$ with $1/\mathfrak{q} = 1/\kappa + 1/6$. Observe that $\mathfrak{q} \in (6/5, 3/2]$ since $\kappa \in (3/2, 2]$. The Sobolev embedding $H^{\tilde{\mathfrak{s}}}(\mathbb{T}^3) \hookrightarrow L^{\mathfrak{q}'}(\mathbb{T}^3)$, with $1/\mathfrak{q}' = 1 - 1/\mathfrak{q} = (5\kappa - 6)/(6\kappa)$ and $\tilde{\mathfrak{s}} \geq 3/\kappa - 1 \in [1/2, 1)$, implies by duality that $L^{\mathfrak{q}}(\mathbb{T}^3) \hookrightarrow H^{-\tilde{\mathfrak{s}}}(\mathbb{T}^3)$. Since $3/\gamma - 1 \geq 3/\kappa - 1$, we can choose $\tilde{\mathfrak{s}} = \mathfrak{s}$. Therefore, the second term of F^ε_1 is bounded in $L^2_{\text{loc}}(\mathbb{R}_+; H^{-\mathfrak{s}-1}(\mathbb{T}^3))$, and we obtain $F^\varepsilon_1 \in L^2_{\text{loc}}(\mathbb{R}_+; (H^{-\mathfrak{s}-1} + W^{-1,3/2} + H^{-1})(\mathbb{T}^3)) \hookrightarrow L^2_{\text{loc}}(\mathbb{R}_+; H^{-\mathfrak{s}-1}(\mathbb{T}^3))$, where the previous injection results from Sobolev embeddings.

We continue with an estimate for F^ε_2. From the continuous embedding $\mathbb{Q}_\perp(L^1(\mathbb{T}^3)) \hookrightarrow W^{-\delta,1}(\mathbb{T}^3)$ which works for all $\delta > 0$, and the energy estimate, the first and fifth terms of F^ε_2 are uniformly bounded in $L^2_{\text{loc}}(\mathbb{R}_+; W^{-1-\delta,1}(\mathbb{T}^3))$. From the energy estimate, the second and third terms of F^ε_2 are uniformly bounded in $L^2_{\text{loc}}(\mathbb{R}_+; W^{-1,1}(\mathbb{T}^3))$. From the following continuous embedding, $\forall \alpha \geq 0$, $\mathbb{Q}_\perp(H^\alpha(\mathbb{T}^3)) \hookrightarrow H^\alpha(\mathbb{T}^3)$, and the energy estimate, the fourth term of F^ε_2 is uniformly bounded in $L^2_{\text{loc}}(\mathbb{R}_+; L^2(\mathbb{T}^3))$. Obviously, from the energy estimate, the last three diffusive terms of F^ε_2 belong to $L^2_{\text{loc}}(\mathbb{R}_+; H^{-1}(\mathbb{T}^3))$. Therefore, we obtain $F^\varepsilon_2 \in L^2_{\text{loc}}(\mathbb{R}_+; (W^{-1-\delta,1} + W^{-1,1} + H^{-1} + L^2)(\mathbb{T}^3)) \hookrightarrow L^2_{\text{loc}}(\mathbb{R}_+; H^{-r}(\mathbb{T}^3))$, with $r > 5/2 + \delta$, by using Sobolev embeddings.

Now, using Lemma 13 of Appendix A, with $\mathfrak{B}_0 = H^{-\mathfrak{s}}(\mathbb{T}^3)$, $\mathfrak{B} = H^{-\sigma}(\mathbb{T}^3)$, $\mathfrak{B}_1 = H^{-r}(\mathbb{T}^3)$, $\mathfrak{s} < \sigma < r$, and $p = q = 2$, we obtain that $\mathcal{U}^\varepsilon$ is compact in $L^2_{\text{loc}}(\mathbb{R}_+; H^{-\sigma}(\mathbb{T}^3))$. We deduce that there exists $\mathcal{U} \equiv {}^t(\psi, {}^t\Psi) \in L^2_{\text{loc}}(\mathbb{R}_+; H^{-\sigma}(\mathbb{T}^3))$ such that $\mathcal{U}^\varepsilon$ converges strongly to $\mathcal{U}$ in $L^2_{\text{loc}}(\mathbb{R}_+; H^{-\sigma}(\mathbb{T}^3))$. Since $\mathbb{P}_\perp \Psi^\varepsilon = 0$ (resp. $\int_{\mathbb{T}^3} dx\, \psi^\varepsilon = 0$), $\forall \varepsilon \geq 0$, then $\mathbb{P}_\perp \Psi = 0$ (resp. $\int_{\mathbb{T}^3} dx\, \psi = 0$). Since the group $\mathcal{S}$ is an isometry in H^α ($\alpha \in \mathbb{R}$), we finally obtain (50).

Step 2. To show that $\mathcal{U} \in L^2_{\text{loc}}(\mathbb{R}_+; L^2(\mathbb{T}^3; \mathbb{R}^3))$, we use the auxiliary variable $\widetilde{U}^\varepsilon := {}^t(\phi^\varepsilon, {}^t\mathbb{Q}_\perp v^\varepsilon_\perp)$. We first establish two points which allow us to deal with a truncated version of $\widetilde{U}^\varepsilon$ instead of $\widetilde{U}^\varepsilon$ itself.

1) From the first statement of Lemma 1 and estimates (42) (which imply, via the De la Vallée Poussin criterion [16], that ϱ^ε is spatially uniformly integrable in $L^{3/2}(\mathbb{T}^3)$, uniformly in time on any compact time interval), we obtain $\|\varrho^\varepsilon - \varrho^\varepsilon \mathbb{1}_{\rho^\varepsilon \leq R}\|_{L^\infty_{\text{loc}}(\mathbb{R}_+; L^\kappa(\mathbb{T}^3))} \to 0$, as $\varepsilon \to 0$, where $R = +\infty$ and $\kappa = 2$, if $\gamma \geq 2$; and where $R \in (3/2, +\infty)$ with R fixed, and $\kappa = \gamma$, if $\gamma < 2$. Moreover, since B^ε is 2-uniformly integrable in space-time, we obtain, from the first statement of Lemma 1, $\|B^\varepsilon_\parallel - B^\varepsilon_\parallel \mathbb{1}_{\rho^\varepsilon \leq R}\|_{L^2_{\text{loc}}(\mathbb{R}_+; L^2(\mathbb{T}^3))} \to 0$, as $\varepsilon \to 0$, for any $R \in (1, +\infty]$. Indeed, the 2-uniform integrability comes from the Gagliardo–Nirenberg interpolation inequality $\|B^\varepsilon\|_{L^{10/3}(\mathbb{R}_+ \times \mathbb{T}^3)} \leq \|B^\varepsilon\|^{2/5}_{L^\infty(\mathbb{R}_+, L^2(\mathbb{T}^3))} \|\nabla B^\varepsilon\|^{3/5}_{L^2(\mathbb{R}_+, L^2(\mathbb{T}^3))} < \infty$ (from uniform bounds of Lemma 2) and the De la Vallée Poussin criterion.

2) From the sixth statement of Lemma 3, we obtain $\|\rho^\varepsilon v^\varepsilon_\perp - v^\varepsilon_\perp\|_{L^2_{\text{loc}}(\mathbb{R}_+; L^q(\mathbb{T}^3))} \to 0$, as $\varepsilon \to 0$, for $1/q = 1/\gamma + 1/6$.

We now set $\widetilde{U}^\varepsilon_R := {}^t(\phi^\varepsilon_R, {}^t\mathbb{Q}_\perp v^\varepsilon_\perp)$, where $\phi^\varepsilon_R := (b^\varepsilon \varrho^\varepsilon + B^\varepsilon_\parallel) \mathbb{1}_{\rho^\varepsilon \leq R}$. Then, from Step 1, and the points 1) to 2) of Step 2, we obtain $\mathcal{S}(-t/\varepsilon) \widetilde{U}^\varepsilon_R \to \mathcal{U}$ in $L^2_{\text{loc}}(\mathbb{R}_+; H^{-\sigma}(\mathbb{T}^3))$−strong. Indeed, we have

$$S(-t/\varepsilon)\widetilde{U}_R^\varepsilon = \mathcal{U} + S(-t/\varepsilon)\,{}^t\big([\phi_R^\varepsilon - \phi^\varepsilon], {}^t\mathbb{Q}_\perp[(1-\rho^\varepsilon)v_\perp^\varepsilon]\big) + \mathcal{S}(-t/\varepsilon)\mathcal{R}^\varepsilon. \tag{54}$$

Since the group $\mathcal{S}$ is an isometry in H^α ($\alpha \in \mathbb{R}$), from the points 1) to 2) of Step 2, the second term of the right-hand side of (54) vanishes as $\varepsilon \to 0$ in $L^2_{\rm loc}(\mathbb{R}_+; (L^\kappa + L^2)(\mathbb{T}^3;\mathbb{R}) \times L^q(\mathbb{T}^3;\mathbb{R}^2))$−strong, and thus in $L^2_{\rm loc}(\mathbb{R}_+; H^{-\sigma}(\mathbb{T}^3))$−strong since $(L^\kappa + L^2)(\mathbb{T}^3) \hookrightarrow H^{-\alpha}(\mathbb{T}^3)$ with $\kappa > 3/2$, and $L^q(\mathbb{T}^3) \hookrightarrow H^{-\sigma}(\mathbb{T}^3)$ with $1/q = 1/\gamma + 1/6$, and where (α, σ) can be chosen such that $1/2 \le \alpha \le \mathfrak{s} \le \sigma < \sigma$. From (50), the third term of the right-hand side of (54) vanishes as $\varepsilon \to 0$ in $L^2_{\rm loc}(\mathbb{R}_+; H^{-\sigma}(\mathbb{T}^3))$−strong.

Finally, the first assertions of Lemma 2 and Lemma 3, and inequality (42) show that $\widetilde{U}_R^\varepsilon$ is bounded in $L^2_{\rm loc}(\mathbb{R}_+; L^2(\mathbb{T}^3;\mathbb{R}^3))$, uniformly with respect to ε. Since the group $\mathcal{S}$ is an isometry in H^α ($\alpha \in \mathbb{R}$), we deduce that $\mathcal{U} \in L^2_{\rm loc}(\mathbb{R}_+; L^2(\mathbb{T}^3;\mathbb{R}^3))$, which ends the proof of the first point of Lemma 4.

We now turn to the proof of the point 2 of Lemma 4. Using the equation (52) on the component Φ^ε, we can see that $\phi^\varepsilon/\varepsilon$ satisfies, $\forall\, \psi_\perp \in H^1_c(\mathbb{R}_+; H^r(\mathbb{T}^3;\mathbb{R}^2))$, with $r > 0$ large enough (specified further),

$$\int_{\mathbb{R}_+} dt \int_{\mathbb{T}^3} dx\, \frac{\phi^\varepsilon}{\varepsilon}\nabla_\perp \cdot \psi_\perp = -\int_{\mathbb{T}^3} dx\, \mathbb{Q}_\perp(\rho_0^\varepsilon v_{0\perp}^\varepsilon)\cdot \psi_\perp(0) - \int_{\mathbb{R}_+} dt \int_{\mathbb{T}^3} dx\,(\rho^\varepsilon - 1)v_\perp^\varepsilon \cdot \mathbb{Q}_\perp \partial_t \psi_\perp$$
$$- \int_{\mathbb{R}_+} dt \int_{\mathbb{T}^3} dx\, v_\perp^\varepsilon \cdot \mathbb{Q}_\perp\partial_t\psi_\perp - \int_{\mathbb{R}_+} dt \int_{\mathbb{T}^3} dx\, F_2^\varepsilon \cdot \psi_\perp \;=\; T_0^\varepsilon + T_1^\varepsilon + T_2^\varepsilon + T_3^\varepsilon\,. \tag{55}$$

From properties of initial conditions (see Section 2.3) we have $\mathbb{Q}_\perp(\rho_0^\varepsilon v_{0\perp}^\varepsilon) \rightharpoonup \mathbb{Q}_\perp u_{0\perp} = \mathbb{Q}_\perp v_{0\perp}$ in $L^{2\gamma/(\gamma+1)}(\mathbb{T}^3)$−weak, and $u_{0\perp} = v_{0\perp} \in L^2(\mathbb{T}^3)$. Defining $\pi_0 := \Delta_\perp^{-1}\nabla_\perp \cdot \mathbb{Q}_\perp u_{0\perp} \in L^2(\mathbb{T}_\parallel; H^1(\mathbb{T}^2_\perp))$, we then have $\nabla_\perp \pi_0 = \mathbb{Q}_\perp u_{0\perp} \in L^2(\mathbb{T}^3)$. From Hölder inequality, we obtain $|T_0^\varepsilon| \le \|\rho_0^\varepsilon v_{0\perp}^\varepsilon\|_{L^{2\gamma/(\gamma+1)}(\mathbb{T}^3)}$ $\|\psi_\perp(0)\|_{L^{2\gamma/(\gamma-1)}(\mathbb{T}^3)}$. This and continuous Sobolev embeddings, imply that there exists a constant $\mathcal{C}_0$ (uniform in ε) such that $|T_0^\varepsilon| \le \mathcal{C}_0\|\psi_\perp\|_{H^1_c(\mathbb{R}_+;H^r(\mathbb{T}^3))}$, with $r > (5/2)^+$. From Hölder inequality we obtain $|T_1^\varepsilon| \le \|\rho^\varepsilon - 1\|_{L^\infty_{\rm loc}(\mathbb{R}_+;L^\gamma(\mathbb{T}^3))}\|v_\perp^\varepsilon\|_{L^2_{\rm loc}(\mathbb{R}_+;L^6(\mathbb{T}^3))}\|\partial_t\mathbb{Q}_\perp\psi_\perp\|_{L^2(\mathbb{R}_+;L^{q'}(\mathbb{T}^3))}$, with $1/q' = 1 - 1/\gamma - 1/6$. Then first, from Lemmas 1 and 3, we obtain $T_1^\varepsilon \to 0$, as $\varepsilon \to 0$. Second, using continuous Sobolev embeddings, there exists a constant $\mathcal{C}_1$ (uniform in ε) such that $|T_1^\varepsilon| \le \mathcal{C}_1\|\psi_\perp\|_{H^1_c(\mathbb{R}_+;H^r(\mathbb{T}^3))}$, with $r > (5/2)^+$. Using Lemma 3 (in particular $\mathbb{Q}_\perp v_\perp = 0$), we obtain $T_2^\varepsilon \to 0$, as $\varepsilon \to 0$. From Hölder inequality, we obtain $|T_2^\varepsilon| \le \|v_\perp^\varepsilon\|_{L^2_{\rm loc}(\mathbb{R}_+;L^2(\mathbb{T}^3))}\|\partial_t\psi_\perp\|_{L^2(\mathbb{R}_+;L^2(\mathbb{T}^3))}$, which implies, together with Lemma 3 and continuous Sobolev embeddings, that there exists a constant $\mathcal{C}_2$ (uniform in ε) such that $|T_2^\varepsilon| \le \mathcal{C}_2\|\psi_\perp\|_{H^1_c(\mathbb{R}_+;H^r(\mathbb{T}^3))}$, with $r > (5/2)^+$. From the Step 1 of this proof, we know that $F_2^\varepsilon \in L^2_{\rm loc}(\mathbb{R}_+; H^{-r}(\mathbb{T}^3))$, for $r > 5/2 + \delta$, and any $\delta > 0$. Since $\nabla_\perp \times F_2^\varepsilon = 0$ in $\mathcal{D}'(\mathbb{R}_+^* \times \mathbb{T}^3)$, there exists $f_2^\varepsilon = \Delta_\perp^{-1}\nabla_\perp \cdot F_2^\varepsilon \in L^2_{\rm loc}(\mathbb{R}_+; H^{-r}(\mathbb{T}^3))$ (uniformly in ε) such that $F_2^\varepsilon = \nabla_\perp f_2^\varepsilon$. Therefore, there exists a function $\pi_1 \in L^2_{\rm loc}(\mathbb{R}_+; H^{-r}(\mathbb{T}^3))$ such that $f_2^\varepsilon \rightharpoonup \pi_1$ in $L^2_{\rm loc}(\mathbb{R}_+; H^{-r}(\mathbb{T}^3))$−weak. Moreover, we deduce that there exists a constant $\mathcal{C}_3$ (uniform in ε) such that $|T_3^\varepsilon| \le \mathcal{C}_3\|\psi_\perp\|_{H^1_c(\mathbb{R}_+;H^r(\mathbb{T}^3))}$, with $r > (5/2)^+$. In summary, we have shown that there exists a constant $\mathcal{C} := \sum_{i=0,\dots,3}\mathcal{C}_i$, uniform in ε, such that for $r > (5/2)^+$,

$$\Big|\int_{\mathbb{R}_+} dt \int_{\mathbb{T}^3} dx\, \nabla_\perp\Big(\frac{\phi^\varepsilon}{\varepsilon}\Big)\cdot\psi_\perp\Big| = \Big|\int_{\mathbb{R}_+} dt \int_{\mathbb{T}^3} dx\, \frac{\phi^\varepsilon}{\varepsilon}\nabla_\perp \cdot \psi_\perp\Big| \le \mathcal{C}\|\psi_\perp\|_{H^1_c(\mathbb{R}_+;H^r(\mathbb{T}^3))}\,,$$

which means that $\nabla_\perp(\phi^\varepsilon/\varepsilon)$ and a fortiori $\phi^\varepsilon/\varepsilon$ belong to $H^{-1}(\mathbb{R}_+; H^{-r}(\mathbb{T}^3))$, uniformly in ε. Moreover, we have proved that

$$\int_{\mathbb{R}_+} dt \int_{\mathbb{T}^3} dx\, \frac{\phi^\varepsilon}{\varepsilon}\nabla_\perp\cdot\psi_\perp \longrightarrow \int_{\mathbb{R}_+} dt \int_{\mathbb{T}^3} dx\, \pi_1\nabla_\perp\cdot\psi_\perp + \int_{\mathbb{T}^3} dx\, \pi_0 \nabla_\perp \cdot \psi_\perp(0)\,.$$

These two properties establish the point 2 of Lemma 4. Using (51), and Lemmas 1 and 2, we obtain that $\phi^\varepsilon \rightharpoonup \phi = b\varrho + B_\parallel = 0$ in $\mathcal{D}'(\mathbb{R}_+^* \times \mathbb{T}^3)$, and $b\varrho + B_\parallel = 0$ in $L^\infty_{\rm loc}(\mathbb{R}_+; L^\kappa(\mathbb{T}^3))$, with $\kappa > 3/2$, which justify the point 3 of Lemma 4. □

We are now able to justify the passage to the limit in equation (21) for $\rho^\varepsilon v_\perp^\varepsilon$. Let us start with the initial condition term. From the discussion about the properties of sequences of initial conditions in Section 2.3 (in particular the uniform bound (27) and the resulting convergences), we can pass to the limit, in the distributional sense, in the initial condition term of (21) to obtain the limit point $u_{0\perp} = v_{0\perp} = \mathbb{P}_\perp u_{0\perp} + \mathbb{Q}_\perp u_{0\perp}$. Using (51) in equation (21), we observe that the term $-\nabla_\perp \pi_0 = -\mathbb{Q}_\perp u_{0\perp}$ (coming from the weak limit of $\phi^\varepsilon/\varepsilon$) cancels the irrotational par $\mathbb{Q}_\perp u_{0\perp}$ of the previous limit point $u_{0\perp}$, so that the limit initial condition is simply $\mathbb{P}_\perp u_{0\perp} = \mathbb{P}_\perp v_{0\perp}$. This is consistent with the fact that in the limit equation the test function $\psi_\perp$ can be chosen divergence-free, i.e., $\mathbb{P}_\perp \psi_\perp = \psi_\perp$. Moreover, according to (6), the two conditions $\nabla_\perp \cdot v_{0\perp} = 0$ (or $\nabla_\perp \cdot u_{0\perp} = 0$, since $u_{0\perp} = v_{0\perp}$) and $B_{0\parallel} + b\varrho_0 = 0$ are related to a preparation of the initial data to avoid fast time oscillations. Since the limit initial condition $\mathbb{P}_\perp u_{0\perp}$ comes naturally without any preparation, then in our framework, we can deal with general data satisfying $\nabla_\perp \cdot u_{0\perp} = \nabla_\perp \cdot v_{0\perp} \not\equiv 0$.

We next deal with the linear terms of (21). Using weak convergence of $v_\perp^\varepsilon$ (resp. B^ε) yielded by the first statement of Lemma 3 (resp. Lemma 2) we can pass to the limit, in the distributional sense, in all linear diffusive terms (resp. the linear advection term $B_\perp^\varepsilon \cdot \partial_\parallel \psi_\perp$) of (21). Using weak convergence for B^ε and strong convergence for $B_\perp^\varepsilon$, which are supplied by Lemma 2, we can pass to the limit in the quadratic nonlinear term $B_\perp^\varepsilon \otimes B^\varepsilon : D_\varepsilon \psi_\perp$ to obtain the term $B_\perp \otimes B_\perp : D_\perp \psi_\perp$. Using the identity $\mathbb{I}_\perp = \mathbb{P}_\perp + \mathbb{Q}_\perp$, for the term involving time derivative in (21), we obtain, $\forall\, \psi_\perp \in \mathscr{C}_c^\infty(\mathbb{R}_+ \times \mathbb{T}^3; \mathbb{R}^2)$,

$$\int_{\mathbb{R}_+} \int_{\mathbb{T}^3} dx\, \rho^\varepsilon v_\perp^\varepsilon \cdot \partial_t \psi_\perp = -T_1^\varepsilon - T_2^\varepsilon + \int_{\mathbb{R}_+} dt \int_{\mathbb{T}^3} dx\, \mathbb{P}_\perp v_\perp^\varepsilon \cdot \partial_t \psi_\perp\,, \tag{56}$$

where the terms T_1^ε and T_2^ε are the same as in (55). For the same reasons as the ones invoked in the proof of the point 2 of Lemma 4, T_1^ε and T_2^ε vanish as $\varepsilon \to 0$. Therefore, using (56) and Lemma 2 (in particular $\mathbb{P}_\perp v_\perp = v_\perp$), we obtain that $\partial_t(\rho^\varepsilon v_\perp^\varepsilon) \rightharpoonup \partial_t v_\perp$ in $\mathcal{D}'(\mathbb{R}_+^* \times \mathbb{T}^3)$. Now in equation (21), we simultaneously deal with the magnetic pressure term $|B^\varepsilon|^2/2$, the singular fluid pressure term $p^\varepsilon/\varepsilon^2$ (with $p^\varepsilon = a(\rho^\varepsilon)^\gamma$), and the singular magnetic term $B_\parallel^\varepsilon/\varepsilon$. Setting $\mathfrak{p}^\varepsilon = p^\varepsilon/\varepsilon^2 + B_\parallel^\varepsilon/\varepsilon + |B^\varepsilon|^2/2$, this term can be rewritten as $\mathfrak{p}^\varepsilon = \phi^\varepsilon/\varepsilon + a(\overline{\rho^\varepsilon})^\gamma/\varepsilon^2 + \pi_2^\varepsilon$, with $\pi_2^\varepsilon = (\gamma-1)\Pi_2(\rho^\varepsilon) + |B^\varepsilon|^2/2$. In $\mathfrak{p}^\varepsilon$, the constant term $a(\overline{\rho^\varepsilon})^\gamma/\varepsilon^2$ is irrelevant because it disappears by spatial integration in (21). From the point 2 of Lemma 4, we obtain $\phi^\varepsilon/\varepsilon \rightharpoonup \pi_1$ in $\mathcal{D}'(\mathbb{R}_+^* \times \mathbb{T}^3)$. In fact, from (51) we have $\phi^\varepsilon/\varepsilon \rightharpoonup \delta_0(t) \otimes \pi_0 + \pi_1$ in $H^{-1}(\mathbb{R}_+, H^{-r}(\mathbb{T}^3))$−weak, but as already mentioned above, the term $\delta_0(t) \otimes \pi_0$ cancels the irrotational part $\mathbb{Q}_\perp u_{0\perp}$ of the limit term $u_{0\perp}$, so that the limit initial condition is $\mathbb{P}_\perp u_{0\perp} = \mathbb{P}_\perp v_{0\perp}$. From energy inequality (30)-(33) with the pressure term Π_2, we obtain $\pi_2^\varepsilon \in L^\infty_{\rm loc}(\mathbb{R}_+; L^1(\mathbb{T}^3))$, uniformly with respect to ε. Then, by weak compactness, there exists a function $\pi_2 \in L^\infty_{\rm loc}(\mathbb{R}_+; L^1(\mathbb{T}^3))$ such that $\pi_2^\varepsilon \rightharpoonup \pi_2$ in $L^\infty_{\rm loc}(\mathbb{R}_+; L^1(\mathbb{T}^3))$−weak−∗. Therefore, we obtain $\mathfrak{p}^\varepsilon \rightharpoonup (\pi_1 + \pi_2)$ in $\mathcal{D}'(\mathbb{R}_+^* \times \mathbb{T}^3)$.

Remark 1. Even if we have the strong convergence of $B_\perp^\varepsilon$ in $L^2_{\rm loc}(\mathbb{R}_+; L^2(\mathbb{T}^3)))$ and a uniform bound in $L^2_{\rm loc}(\mathbb{R}_+; H^1(\mathbb{T}^3))$ for B^ε, we do not have $|B^\varepsilon|^2 \rightharpoonup |B|^2$ in $\mathcal{D}'(\mathbb{R}_+^* \times \mathbb{T}^3)$. The reason of this lack of convergence comes from the fact that $\partial_t B_\parallel$ is not bounded, uniformly with respect to ε, in some suitable functional spaces. Indeed, equation (24) for $B_\parallel^\varepsilon$ contains a singular term in $1/\varepsilon$, which prevents such a boundedness. In other words, this is fast time oscillations in the parallel direction that prevent time compactness and thus, such a convergence.

It remains to pass to the limit in the nonlinear term $\rho^\varepsilon v_\perp^\varepsilon \otimes v^\varepsilon$ in (21). This is the purpose of the following lemma.

Lemma 5. *There exists a distribution* $\pi_3 \in \mathcal{D}'(\mathbb{R}_+^* \times \mathbb{T}^3)$, *such that*

$$\nabla_\varepsilon \cdot (\rho^\varepsilon v_\perp^\varepsilon \otimes v^\varepsilon) \longrightarrow \nabla_\perp \cdot (v_\perp \otimes v_\perp) \ + \ \nabla_\perp \pi_3 \quad in \ \ \mathcal{D}'(\mathbb{R}_+^* \times \mathbb{T}^3)\,.$$

Using Lemma 5, we can complete the justification of the passage to the limit, in the distributional sense, in equations (21) to obtain the second equation of (16) in the sense of distributions.

Remark 2. In (12) the pressure term π, which can be seen as a Lagrange multiplier ensuring the constraint $\nabla_\perp \cdot v_\perp = 0$, comes from the following three contributions, π_1, π_2 and π_3. The pressure term π_1 comes from the singular fluid pressure term and the singular magnetic term. The pressure π_2 comes from the non-singular fluid and magnetic pressure terms. The pressure term π_3, which comes from the Reynolds stress tensor, results from taking into account the resonant interactions of the compressible modes on the incompressible mode, while the non-resonant interaction terms vanish in the limit by using Riemann–Lebesgue or stationary phase arguments.

Proof of Lemma 5. Observe first the following decomposition, $\forall\, \psi_\perp \in \mathscr{C}_c^\infty(\mathbb{R}_+ \times \mathbb{T}^3; \mathbb{R}^2)$,

$$\begin{aligned} \int_{\mathbb{R}_+} dt \int_{\mathbb{T}^3} dx\, \rho^\varepsilon v_\perp^\varepsilon \otimes v^\varepsilon : D_\varepsilon \psi_\perp = & \int_{\mathbb{R}_+} dt \int_{\mathbb{T}^3} dx\, \rho^\varepsilon v_\perp^\varepsilon \otimes v_\perp^\varepsilon : D_\perp \psi_\perp \\ & + \varepsilon \int_{\mathbb{R}_+} dt \int_{\mathbb{T}^3} dx\, \rho^\varepsilon v_\parallel^\varepsilon v_\perp^\varepsilon \cdot \partial_\parallel \psi_\perp \ = \bar{\Gamma}_1^\varepsilon + \bar{\Gamma}_2^\varepsilon\,. \end{aligned} \tag{57}$$

Using Hölder inequality, we obtain $|\bar{\Gamma}_2^\varepsilon| \leq \varepsilon \|\rho^\varepsilon |v^\varepsilon|^2\|_{L^\infty_{\rm loc}(\mathbb{R}_+;L^1(\mathbb{T}^3))} \|\partial_\parallel \psi_\perp\|_{L^1(\mathbb{R}_+;L^\infty(\mathbb{T}^3))}$. Therefore, exploiting the energy estimate, we have $\bar{\Gamma}_2^\varepsilon \to 0$, as $\varepsilon \to 0$.

To deal with the term $\bar{\Gamma}_1^\varepsilon$, we follow the spirit of the proof of the convergence result in the part III of [34]. For this we consider the following decomposition

$$\begin{aligned} \Gamma_1^\varepsilon := \nabla_\perp \cdot (\rho^\varepsilon v_\perp^\varepsilon \otimes v_\perp^\varepsilon) = \sum_{i=1}^5 \Gamma_{1i}^\varepsilon = \nabla_\perp \cdot \big(& \rho^\varepsilon v_\perp^\varepsilon \otimes \mathbb{P}_\perp v_\perp^\varepsilon \ + \mathbb{P}_\perp(\rho^\varepsilon v_\perp^\varepsilon) \otimes \mathbb{Q}_\perp v_\perp^\varepsilon \\ & + (\mathbb{Q}_\perp[\rho^\varepsilon v_\perp^\varepsilon] - \Phi^\varepsilon) \otimes \mathbb{Q}_\perp v_\perp^\varepsilon \ + \Phi^\varepsilon \otimes (\mathbb{Q}_\perp v_\perp^\varepsilon - \Phi^\varepsilon) + \Phi^\varepsilon \otimes \Phi^\varepsilon \big)\,, \end{aligned} \tag{58}$$

where we set $\Phi^\varepsilon := \mathcal{S}_2(t/\varepsilon)\mathcal{U}$. We successively deal with the terms Γ_{1i}^ε, for $i \in \{1, \ldots, 5\}$. We start with Γ_{11}^ε. Using the fourth and fifth assertions of Lemma 3 with $s = \sigma \in [\mathfrak{s}, 1)$, we obtain $\Gamma_{11}^\varepsilon \rightharpoonup \nabla_\perp \cdot (v_\perp \otimes v_\perp)$ in $\mathcal{D}'(\mathbb{R}_+^* \times \mathbb{T}^3)$, which gives the first part of the limit in Lemma 5. We continue with the term Γ_{12}^ε. Using the third and seventh assertions of Lemma 3, since $1/q + 1/6 < 1$, we obtain $\Gamma_{12}^\varepsilon \rightharpoonup 0$ in $\mathcal{D}'(\mathbb{R}_+^* \times \mathbb{T}^3)$. For the term Γ_{13}^ε, we observe that $\mathbb{Q}_\perp[\rho^\varepsilon v_\perp^\varepsilon] - \Phi^\varepsilon = \mathcal{R}_2^\varepsilon$, where the term $\mathcal{R}_2^\varepsilon \in \mathbb{R}^2$ is the second component of the error term $\mathcal{R}^\varepsilon = {}^t(\mathcal{R}_1^\varepsilon, {}^t\mathcal{R}_2^\varepsilon)$ involved in equation (50) of Lemma 4. Using (50) with $\sigma = 1$ and the first or the third statement of Lemma 3, we obtain $\Gamma_{13}^\varepsilon \rightharpoonup 0$ in $\mathcal{D}'(\mathbb{R}_+^* \times \mathbb{T}^3)$. We pursue with the term Γ_{14}^ε. Let Φ_η^ε be a regularization of Φ^ε obtained by the following way. Using the fact that $\mathscr{C}_c^\infty$ is dense in L^2, we define $\Phi_\eta^\varepsilon := \mathcal{S}_2(t/\varepsilon)\mathcal{U}_\eta$, where $\mathcal{U}_\eta \in \mathscr{C}_c^\infty(\mathbb{R}_+ \times \mathbb{T}^3)$ is such that $\|\mathcal{U}_\eta - \mathcal{U}\|_{L^2_{\rm loc}(\mathbb{R}_+;L^2(\mathbb{T}^3))} \leq \eta$, with $0 \leq \eta \ll 1$. We consider the decomposition

$$\Phi^\varepsilon \otimes (\mathbb{Q}_\perp v_\perp^\varepsilon - \Phi^\varepsilon) = (\Phi_\eta^\varepsilon - \Phi^\varepsilon) \otimes (\mathbb{Q}_\perp v_\perp^\varepsilon - \Phi^\varepsilon) + \Phi_\eta^\varepsilon \otimes (\mathbb{Q}_\perp v_\perp^\varepsilon - \Phi^\varepsilon) =: R_{1\eta}^\varepsilon + R_{2\eta}^\varepsilon\,. \tag{59}$$

For the term $R_{1\eta}^\varepsilon$, using the isometry property of $\mathcal{S}$, there exists a constant $C = C(\|\mathcal{U}\|_{L^2_{\rm loc}(\mathbb{R}_+;L^2(\mathbb{T}^3))})$ such that $\|R_{1\eta}^\varepsilon\|_{L^1_{\rm loc}(\mathbb{R}_+;L^1(\mathbb{T}^3))} \leq C\eta$. For the term $R_{2\eta}^\varepsilon$, using the isometry of $\mathcal{S}$, we first observe that $\Phi_\eta^\varepsilon \in L^2_{\rm loc}(\mathbb{R}_+; H^m(\mathbb{T}^3))$, with $m \geq 0$, and for all $\eta > 0$. Second, we claim that $\mathbb{Q}_\perp v_\perp^\varepsilon - \Phi^\varepsilon \to 0$ in

$L^2_{\mathrm{loc}}(\mathbb{R}_+; H^{-\sigma}(\mathbb{T}^3))$−strong, for $\mathfrak{s} < \sigma < (5/2)^+$. Indeed, $\mathbb{Q}_\perp v^\varepsilon_\perp - \Phi^\varepsilon = \mathcal{R}^\varepsilon_2 + \mathbb{Q}_\perp((1-\rho^\varepsilon)v^\varepsilon_\perp)$, where, using (50), we have $\mathcal{R}^\varepsilon_2 \to 0$ in $L^2_{\mathrm{loc}}(\mathbb{R}_+; H^{-\sigma}(\mathbb{T}^3))$−strong. From the sixth statement of Lemma 3, the continuity of $\mathbb{Q}_\perp$ in L^q with $1/q = 1/\gamma + 1/6$, and the embedding $L^q(\mathbb{T}^3) \hookrightarrow H^{-\sigma}(\mathbb{T}^3)$ for $\sigma \geq \mathfrak{s}$ (cf. proof of Lemma 3), we obtain $\mathbb{Q}_\perp((1-\rho^\varepsilon)v^\varepsilon_\perp) \to 0$ in $L^2_{\mathrm{loc}}(\mathbb{R}_+; H^{-\sigma}(\mathbb{T}^3))$−strong, for all $\sigma \geq \sigma$. Therefore, in the right-hand side of (59), taking first the limit $\varepsilon \to 0$ and then the limit $\eta \to 0$, we obtain $\Gamma^\varepsilon_{14} \rightharpoonup 0$ in $\mathcal{D}'(\mathbb{R}^*_+ \times \mathbb{T}^3)$. It remains to show that $\Gamma^\varepsilon_{15} \rightharpoonup \pi_3$ in $\mathcal{D}'(\mathbb{R}^*_+ \times \mathbb{T}^3)$. For this, using Fourier series, we can compute explicitly the term Γ^ε_{15}. We set $\mathfrak{U}^\varepsilon \equiv {}^t(\phi^\varepsilon, {}^t\Phi^\varepsilon) := {}^t(\mathcal{S}_1(t/\varepsilon)\mathcal{U}, {}^t\mathcal{S}_2(t/\varepsilon)\mathcal{U}) = \mathcal{S}(t/\varepsilon)\mathcal{U}$, with $\mathcal{U} = {}^t(\psi, {}^t\Psi) \in L^2_{\mathrm{loc}}(\mathbb{R}_+; L^2(\mathbb{T}^3))$, and such that $\int_{\mathbb{T}^3} dx\, \psi = 0$, and $\mathbb{P}_\perp \Psi = 0$. Since $\mathbb{P}_\perp \Psi = 0$, then $\Psi = \nabla_\perp \psi$, with $\psi = \Delta_\perp^{-1} \nabla_\perp \cdot \Psi \in L^2_{\mathrm{loc}}(\mathbb{R}_+; L^2(\mathbb{T}_\parallel; H^1(\mathbb{T}^2_\perp)))$. This regularity is deduced from the L^2 regularity of $\mathcal{U}$. Similarly, since $\mathcal{S}$ and $\mathbb{P}_\perp$ commute, we obtain $\int_{\mathbb{T}^3} dx\, \phi^\varepsilon = 0$, and $\mathbb{P}_\perp \Phi^\varepsilon = 0$, so that $\Phi^\varepsilon = \nabla_\perp \varphi^\varepsilon$, with $\varphi^\varepsilon = \Delta_\perp^{-1} \nabla_\perp \cdot \Phi^\varepsilon$. We introduce the Fourier series

$$\psi = \sum_{k \in \mathbb{Z}^3} \psi_k(t) e^{\mathrm{i}k\cdot x}, \quad \Psi = \mathrm{i} \sum_{k \in \mathbb{Z}^3} k_\perp \psi_k(t) e^{\mathrm{i}k\cdot x},$$

with $\psi_0(t) = 0$, and

$$\|\{\psi_k\}_k\|_{L^2_{\mathrm{loc}}(\mathbb{R}_+;\ell^2(\mathbb{Z}^3))} + \|\{\psi_k |k_\perp|\}_k\|_{L^2_{\mathrm{loc}}(\mathbb{R}_+;\ell^2(\mathbb{Z}^3))} =: \mathcal{N}_0 < \infty, \tag{60}$$

where the last estimate comes from the L^2 regularity of $\mathcal{U}$ stated in Lemma 4. We denote by $\mathfrak{U}^\varepsilon_k = {}^t(\phi^\varepsilon_k, {}^t\Phi^\varepsilon_k = \mathrm{i}\, {}^t k_\perp \varphi^\varepsilon_k)$ the Fourier coefficients of $\mathfrak{U}^\varepsilon \equiv {}^t(\phi^\varepsilon, {}^t\Phi^\varepsilon = \nabla_\perp \varphi^\varepsilon)$. Inserting the Fourier series of $\mathfrak{U}^\varepsilon$ in the linear equation $\partial_t \mathfrak{U}^\varepsilon = \mathcal{L}\mathfrak{U}^\varepsilon/\varepsilon$, we are led to solve linear second-order ODEs in time for the Fourier coefficients $\phi^\varepsilon_k(t)$ and $\varphi^\varepsilon_k(t)$, with the inital conditions $\mathfrak{U}^\varepsilon_k(0) = \mathcal{U}_k(t)$ and $\partial_t \mathfrak{U}^\varepsilon_k(0) = \mathcal{L}_k \mathcal{U}_k(t)/\varepsilon$, where $\mathcal{L}_k = \mathrm{i}\, {}^t(-c^\varepsilon k_\perp \cdot, {}^t k_\perp)$. Solving these linear ODEs, we obtain for Φ^ε,

$$\Phi^\varepsilon = \nabla_\perp \varphi^\varepsilon = \mathrm{i} \sum_{k \in \mathbb{Z}^3} e^{\mathrm{i}k\cdot x}\, k_\perp \left\{ \psi_k(t) \cos\left(\sqrt{c^\varepsilon}|k_\perp|\tfrac{t}{\varepsilon}\right) - \tfrac{1}{\sqrt{c^\varepsilon}|k_\perp|} \psi_k(t) \sin\left(\sqrt{c^\varepsilon}|k_\perp|\tfrac{t}{\varepsilon}\right) \right\}.$$

We then obtain

$$\Phi^\varepsilon \otimes \Phi^\varepsilon = - \sum_{k,l \in \mathbb{Z}^3} e^{\mathrm{i}(k+l)\cdot x}\, \theta^\varepsilon_k(t)\theta^\varepsilon_l(t)(k_\perp \otimes l_\perp) = -\big(S^\varepsilon_1(t,x) + S^\varepsilon_2(t,x)\big), \tag{61}$$

with $\theta^\varepsilon_k(t) = \psi_k(t)\cos(\sqrt{c^\varepsilon}|k_\perp|t/\varepsilon) - \big(\psi_k(t)/(\sqrt{c^\varepsilon}|k_\perp|)\big)\sin(\sqrt{c^\varepsilon}|k_\perp|t/\varepsilon) \in L^2_{\mathrm{loc}}(\mathbb{R}_+)$, $\forall\, k \in \mathbb{Z}^3$, and where S^ε_1 (resp. S^ε_2) is the sum of (k,l) on the subset $\Lambda_1 = \{(k,l) \in \mathbb{Z}^3 \times \mathbb{Z}^3 \;;\; |k_\perp| \neq |l_\perp|\}$ (resp. $\Lambda_2 = \{(k,l) \in \mathbb{Z}^3 \times \mathbb{Z}^3 \;;\; |k_\perp| = |l_\perp|\}$). We next show that $S^\varepsilon_1 \rightharpoonup 0$ in $\mathcal{D}'(\mathbb{R}^*_+ \times \mathbb{T}^3)$, and that $\nabla_\perp \cdot S^\varepsilon_2$ is a perpendicular gradient. We first deal with S^ε_1. Let $\varphi(t,x) = \chi(t)\lambda(x) \in \mathscr{C}^\infty_c(\mathbb{R}_+ \times \mathbb{T}^3)$. Then we obtain,

$$\langle S^\varepsilon_1, \varphi \rangle = |\mathbb{T}^3| \sum_{(k,l)\in\Lambda_1} \lambda_{k+l}\,(k_\perp \otimes l_\perp) \int_{\mathbb{R}_+} dt\, \theta^\varepsilon_k(t)\theta^\varepsilon_l(t)\chi(t) = \sum_{(k,l)\in\Lambda_1} \mathcal{H}^\varepsilon_{kl}, \tag{62}$$

where λ_k is the Fourier coefficients of λ. Using Cauchy–Scwharz inequality in time, we obtain $|\mathcal{H}^\varepsilon_{kl}| \leq |\mathbb{T}^3| \|\chi\|_{L^\infty(\mathbb{R}_+)} d_k d_l |\lambda_{k+l}|$, with $d_k := (|k_\perp| \|\psi_k\|_{L^2_{\mathrm{loc}}(\mathbb{R}_+)} + \|\psi_k\|_{L^2_{\mathrm{loc}}(\mathbb{R}_+)}) \max\{1, 1/\sqrt{\underline{c}}\} \geq |k_\perp|\, \|\theta^\varepsilon_k\|_{L^2_{\mathrm{loc}}(\mathbb{R}_+)}$. Using this estimate, bound (60), and Cauchy–Schwarz inequality for one of the infinite sums in (62), we obtain $|\langle S^\varepsilon_1, \varphi \rangle| \leq 2\mathcal{N}^2_0 |\mathbb{T}^3|\, \|\chi\|_{L^\infty(\mathbb{R}_+)} \max\{1, 1/\sqrt{\underline{c}}\}^2 \sum_{k \in \mathbb{Z}^3} |\lambda_k|$. This last sum converges because, from the regularity of λ, the Fourier coefficients λ_k decrease enough with respect to k. Then S^ε_1 is summable in $\mathcal{D}'(\mathbb{R}^*_+ \times \mathbb{T}^3)$. Now, using the Riemann–Lebesgue theorem and the condition $|k_\perp| \neq |l_\perp|$ for $(k,l) \in \Lambda_1$, recombining the oscillating products involving cos and sin, we easily show that $\theta^\varepsilon_k \theta^\varepsilon_l \rightharpoonup 0$ in $L^1_{\mathrm{loc}}(\mathbb{R}_+)$−weak. Using this vanishing limit and the summability of S^ε_1, we obtain $S^\varepsilon_1 \rightharpoonup 0$, in $\mathcal{D}'(\mathbb{R}^*_+ \times \mathbb{T}^3)$. We next deal

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with S_2^ε. With the same arguments as those used for S_1^ε, S_2^ε is summable in $\mathcal{D}'(\mathbb{R}_+^* \times \mathbb{T}^3)$. It remains to show that $\nabla_\perp \cdot S_2^\varepsilon$ is a perpendicular gradient. Symmetrizing the sum S_2^ε in (k,l) (such that the expression of the general term remains invariant by exchanging k and l), using the change of variable $l = n - k$ in S_2^ε, and applying the divergence operator $\nabla_\perp \cdot$ to S_2^ε, we obtain

$$\nabla_\perp \cdot S_2^\varepsilon = \frac{\mathrm{i}}{2} \sum_{n \in \mathbb{Z}^3} e^{\mathrm{i} n \cdot x} \sum_{|k_\perp| = |n_\perp - k_\perp|} \{(k_\perp \otimes [n_\perp - k_\perp]) + ([n_\perp - k_\perp] \otimes k_\perp)\}\, n_\perp \theta_n^\varepsilon \theta_{n-k}^\varepsilon \, .$$

To simplify this expression, we first observe that $|k_\perp| = |n_\perp - k_\perp|$ implies $|n_\perp|^2 = 2 n_\perp \cdot k_\perp$. Using this identity, we obtain $(k_\perp \otimes [n_\perp - k_\perp]) + ([n_\perp - k_\perp] \otimes k_\perp) = n_\perp (n_\perp \cdot k_\perp) = n_\perp |n_\perp|^2/2$ so that

$$\nabla_\perp \cdot S_2^\varepsilon = \nabla_\perp \Big(\frac{1}{4} \sum_{n \in \mathbb{Z}^3} |n_\perp|^2 e^{\mathrm{i} n \cdot x} \sum_{k \in \mathbb{Z}^3} \theta_n^\varepsilon \theta_{n-k}^\varepsilon \Big) =: -\nabla_\perp \pi_3^\varepsilon \longrightarrow -\nabla_\perp \pi_3 \quad \text{in } \mathcal{D}'(\mathbb{R}_+^* \times \mathbb{T}^3) \, .$$

This ends the proof of Lemma 5. □

4.6. Passage to the limit in the equation for $\rho^\varepsilon v_\parallel^\varepsilon$

Here, we justify the passage to the limit in equation (22) for $\rho^\varepsilon v_\parallel^\varepsilon$. Let us start with the initial condition term. From the discussion about the properties of sequences of initial conditions in Section 2.3 (in particular the uniform bound (27) and the resulting convergences), we can pass to the limit, in the distributional sense, in the initial condition term of (22) to obtain the limit initial condition $u_{0\parallel} = v_{0\parallel}$ (since $u_0 = v_0$). Next, using weak convergence of $v_\parallel^\varepsilon$ yielded by the first statement of Lemma 3, we can pass to the limit, in the distributional sense, in all linear diffusive terms of (22). Using the same arguments as those used to deal with (56) and show that $\partial_t(\rho^\varepsilon v_\perp^\varepsilon) \rightharpoonup \partial_t v_\perp$ in $\mathcal{D}'(\mathbb{R}_+^* \times \mathbb{T}^3)$, we obtain $\partial_t(\rho^\varepsilon v_\parallel^\varepsilon) \rightharpoonup \partial_t v_\parallel$ in $\mathcal{D}'(\mathbb{R}_+^* \times \mathbb{T}^3)$. Regarding the terms $\varepsilon \lambda^\varepsilon v^\varepsilon \cdot \nabla_\varepsilon \partial_\parallel \psi_\parallel$ and $\varepsilon(|B^\varepsilon|/2)\partial_\parallel \psi_\parallel$, uniform bounds in $L^2_{\mathrm{loc}}(\mathbb{R}_+; L^2(\mathbb{T}^3))$ for v^ε and B^ε, given by the energy estimate, show that the terms $\varepsilon \lambda^\varepsilon \partial_\parallel \nabla_\varepsilon \cdot v^\varepsilon$ and $\varepsilon \partial_\parallel (|B^\varepsilon|/2)$ vanish in $\mathcal{D}'(\mathbb{R}_+^* \times \mathbb{T}^3)$, as $\varepsilon \to 0$. For the term $B_\parallel^\varepsilon \otimes B^\varepsilon : D_\varepsilon \psi_\parallel$, we consider the following decomposition, $\forall\, \psi_\parallel \in \mathscr{C}_c^\infty(\mathbb{R}_+ \times \mathbb{T}^3; \mathbb{R})$,

$$\int_{\mathbb{R}_+} dt \int_{\mathbb{T}^3} dx\, B_\parallel^\varepsilon \otimes B^\varepsilon : D_\varepsilon \psi_\parallel = \int_{\mathbb{R}_+} dt \int_{\mathbb{T}^3} dx\, B_\parallel^\varepsilon B_\perp^\varepsilon \cdot \nabla_\perp \psi_\parallel \; + \; \varepsilon \int_{\mathbb{R}_+} dt\, |B_\parallel^\varepsilon|^2 \partial_\parallel \psi_\parallel \, . \tag{63}$$

Using the uniform bound in $L^2_{\mathrm{loc}}(\mathbb{R}_+; L^2(\mathbb{T}^3))$ for B^ε, given by the energy estimate, the second term of (63) vanishes as $\varepsilon \to 0$. Using weak convergence of $B_\parallel^\varepsilon$ and strong convergence of $B_\perp^\varepsilon$, given respectively by the first and third statements of Lemma 2, we obtain $\nabla_\perp \cdot (B_\parallel^\varepsilon B_\perp^\varepsilon) \rightharpoonup \nabla_\perp \cdot (B_\parallel B_\perp)$ in $\mathcal{D}'(\mathbb{R}_+^* \times \mathbb{T}^3)$, which ends the treatment of the first term of (63). Therefore, we obtain $\nabla_\varepsilon \cdot (B_\parallel^\varepsilon \otimes B^\varepsilon) \rightharpoonup \nabla_\perp \cdot (B_\parallel B_\perp)$ in $\mathcal{D}'(\mathbb{R}_+^* \times \mathbb{T}^3)$. For the singular fluid pressure term $p^\varepsilon/\varepsilon$, we rewrite this term as $p^\varepsilon/\varepsilon = \varepsilon(\gamma - 1)\Pi_2(\rho^\varepsilon) + b^\varepsilon \varrho^\varepsilon + a(\overline{\rho^\varepsilon})^\gamma/\varepsilon$. The constant term $a(\overline{\rho^\varepsilon})^\gamma/\varepsilon$ is irrelevant because it disappears by spatial integration in (22). From energy inequality (30)-(33) with the pressure term Π_2, we obtain $\varepsilon(\gamma-1)\Pi_2(\rho^\varepsilon) \to 0$ in $L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^1(\mathbb{T}^3))$−strong. From the second statement of Lemma 1, we obtain $b^\varepsilon \varrho^\varepsilon \rightharpoonup b \varrho$ in $L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^\kappa(\mathbb{T}^3))$−weak−∗. Therefore, we obtain $\partial_\parallel (p^\varepsilon/\varepsilon) \rightharpoonup b \partial_\parallel \varrho = -\partial_\parallel B_\parallel$ in $\mathcal{D}'(\mathbb{R}_+^* \times \mathbb{T}^3)$, where for the last equality we have used the point 3 of Lemma 4. It remains to deal with the term $\rho^\varepsilon v_\parallel^\varepsilon \otimes v^\varepsilon : D_\varepsilon \psi_\parallel$, for which we consider the following decomposition, $\forall\, \psi_\parallel \in \mathscr{C}_c^\infty(\mathbb{R}_+ \times \mathbb{T}^3; \mathbb{R})$,

$$\int_{\mathbb{R}_+} dt \int_{\mathbb{T}^3} dx\, \rho^\varepsilon v_\parallel^\varepsilon \otimes v^\varepsilon : D_\varepsilon \psi_\parallel = \int_{\mathbb{R}_+} dt \int_{\mathbb{T}^3} dx\, \rho^\varepsilon v_\parallel^\varepsilon v_\perp^\varepsilon \cdot \nabla_\perp \psi_\parallel \; + \; \varepsilon \int_{\mathbb{R}_+} dt \int_{\mathbb{T}^3} dx\, \rho^\varepsilon |v_\parallel^\varepsilon|^2 \partial_\parallel \psi_\parallel \, . \tag{64}$$

Since from the energy estimate, $\|\rho^\varepsilon |v^\varepsilon|^2\|_{L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^1(\mathbb{T}^3))}$ is bounded uniformly with respect to ε, the second term of (64) vanishes as $\varepsilon \to 0$. Finally, it remains to deal with the first term of (64). This term is the most delicate, because we have only weak compactness for $\rho^\varepsilon v_\parallel^\varepsilon$ and $v_\perp^\varepsilon$. Indeed, even if $\mathbb{P}_\perp v_\perp^\varepsilon$ converge strongly, $\mathbb{Q}_\perp v_\perp^\varepsilon$ converge weakly (to zero, see Lemma 3). Therefore, to pass to the limit in this term, we will use Lemma 14 of Appendix A, for which we show below that its hypotheses, splitted in three points, are satisfied. 1) From the fifth statement of Lemma 3, we obtain $\rho^\varepsilon v_\parallel^\varepsilon \rightharpoonup v_\parallel$ in $L^2_{\rm loc}(\mathbb{R}_+; L^{6\gamma/(6+\gamma)}(\mathbb{T}^3))$−weak. 2) The uniform bound $v_\perp^\varepsilon \in L^2_{\rm loc}(\mathbb{R}_+; L^6(\mathbb{T}^3))$ and Lemma 4.3 in [7] imply $\|v_\perp^\varepsilon(t,\cdot+h) - v_\perp^\varepsilon(t,\cdot)\|_{L^2_{\rm loc}(\mathbb{R}_+;L^6(\mathbb{T}^3))} \to 0$, as $|h| \to 0$, uniformly with respect to ε. 3) From equation (22) for $\rho^\varepsilon v_\parallel^\varepsilon$, we obtain in $\mathcal{D}'(\mathbb{R}_+^* \times \mathbb{T}^3)$,

$$\partial_t(\rho^\varepsilon v_\parallel^\varepsilon) = -\nabla_\varepsilon \cdot (\rho^\varepsilon v_\parallel^\varepsilon \otimes v^\varepsilon) + \partial_\parallel \big(\tfrac{1}{\varepsilon} p^\varepsilon + \tfrac{\varepsilon}{2}|B^\varepsilon|^2\big) + \nabla_\varepsilon \cdot (B_\parallel^\varepsilon \otimes B^\varepsilon) + \mu_\perp^\varepsilon \Delta_\perp v_\parallel^\varepsilon + \mu_\parallel^\varepsilon \Delta_\parallel v_\parallel^\varepsilon + \varepsilon\lambda^\varepsilon \partial_\parallel \nabla_\varepsilon \cdot v^\varepsilon \,. \quad (65)$$

Using the energy estimate and the preceding bound for the pressure term $p^\varepsilon/\varepsilon$ (already used in this section), we obtain $\partial_t(\rho^\varepsilon v_\parallel^\varepsilon) \in L^\infty_{\rm loc}(\mathbb{R}_+; (W^{-1,1} + W^{-1,\kappa})(\mathbb{T}^3)) + L^2_{\rm loc}(\mathbb{R}_+; H^{-1}(\mathbb{T}^3)) \hookrightarrow L^1_{\rm loc}(\mathbb{R}_+; W^{-1,1}(\mathbb{T}^3))$. Gathering points 1) to 3), we can apply Lemma 14 of Appendix A with $g^\varepsilon = \rho^\varepsilon v_\parallel^\varepsilon$, $h^\varepsilon = v_\perp^\varepsilon$, $p_1 = q_1 = 2$ $(1/p_1 + 1/q_1 = 1)$, $p_2 = 6\gamma/(6+\gamma)$, and $q_2 = 6$ $(1/p_2 + 1/q_2 = 1/\gamma + 1/3 < 1$, for $\gamma > 3/2)$, to obtain $\nabla_\perp \cdot (\rho^\varepsilon v_\parallel^\varepsilon v_\perp^\varepsilon) \rightharpoonup \nabla_\perp \cdot (v_\parallel v_\perp)$ in $\mathcal{D}'(\mathbb{R}_+^* \times \mathbb{T}^3)$. In conclusion, we have shown that the weak formulation (22) converges to the second equation of (16) in the sense of distributions.

4.7. Passage to the limit in a combination of the equations for ϱ^ε and $B_\parallel^\varepsilon$

The passage to the limit in the equation of $B_\parallel^\varepsilon$ is more delicate, because, unlike what is done to treat the asymptotic limit in the perpendicular direction, here, we can not use the unitary group method to deal with the singular term in $1/\varepsilon$ in equation (24) for $B_\parallel^\varepsilon$. From the study of the asymptotic limit in the perpendicular direction, more precisely the point 3 of the Lemma 4, we observe a relation between $B_\parallel$ and ϱ, namely, $B_\parallel + b\varrho = B_\parallel + \mathbb{p} = 0$, where we define $\mathbb{p} = a\gamma\varrho$. From this relation, the idea is to cancel the $1/\varepsilon$-singularity in equation (24) for $B_\parallel^\varepsilon$ with the $1/\varepsilon$-singularity coming from the equation of ϱ^ε, this latter equation being obtained from equation (20) for ρ^ε. Indeed, from equation (20), we construct the following equation for $\varrho^\varepsilon/\overline{\varrho^\varepsilon} = (\rho^\varepsilon/\overline{\varrho^\varepsilon} - 1)/\varepsilon$, $\forall \varphi \in \mathscr{C}_c^\infty(\mathbb{R}_+ \times \mathbb{T}^3; \mathbb{R})$,

$$\int_\Omega dx\, \frac{\varrho_0^\varepsilon}{\overline{\rho_0^\varepsilon}}\varphi(0) + \int_0^\infty dt \int_\Omega dx \left(\frac{\varrho^\varepsilon}{\overline{\rho^\varepsilon}}(\partial_t + v_\perp^\varepsilon \cdot \nabla_\perp)\varphi + \frac{1}{\varepsilon} v_\perp^\varepsilon \cdot \nabla_\perp \varphi + \frac{\rho^\varepsilon}{\overline{\rho^\varepsilon}} v_\parallel^\varepsilon \partial_\parallel \varphi \right) = 0 \,. \quad (66)$$

Let us define the auxiliary component

$$\mathbb{B}_\parallel^\varepsilon := \frac{1}{\mathbb{c}} \left(B_\parallel^\varepsilon - \frac{\varrho^\varepsilon}{\overline{\rho^\varepsilon}} \right), \qquad \mathbb{B}_{0\parallel}^\varepsilon := \frac{1}{\mathbb{c}} \left(B_{0\parallel}^\varepsilon - \frac{\varrho_0^\varepsilon}{\overline{\rho_0^\varepsilon}} \right), \qquad \mathbb{c} = 1 + \frac{1}{b} \,.$$

Taking $\varphi = \psi_\parallel$ in (66), where $\psi_\parallel$ is the same test function as the one used in equation (24), and substracting to (24), we obtain

$$\mathbb{c} \int_\Omega dx\, \mathbb{B}_{0\parallel}^\varepsilon \psi_\parallel(0) + \mathbb{c} \int_0^\infty dt \int_\Omega dx\, \big(\mathbb{B}_\parallel^\varepsilon (\partial_t + v_\perp^\varepsilon \cdot \nabla_\perp)\psi_\parallel$$
$$- v_\parallel^\varepsilon B_\perp^\varepsilon \cdot \nabla_\perp \psi_\parallel - \frac{\rho^\varepsilon}{\overline{\rho^\varepsilon}} v_\parallel^\varepsilon \partial_\parallel \psi_\parallel + \eta_\perp B_\parallel^\varepsilon \Delta_\perp \psi_\parallel + \eta_\parallel B_\parallel^\varepsilon \Delta_\parallel \psi_\parallel \big) = 0 \,. \quad (67)$$

In the sense of distributions, this reveals slow dynamics on $\mathbb{B}_\parallel^\varepsilon$. Since $B_\parallel + b\varrho = 0$, the weak limit of $\mathbb{B}_\parallel^\varepsilon$ is $\mathbb{B}_\parallel = (B_\parallel - \varrho)/\mathbb{c} = B_\parallel$. This means that, after a boundary layer near $t = 0$, the asymptotic behavior of $\mathbb{B}_\parallel^\varepsilon$ is similar to the one of $B_\parallel$. And, because $B_\parallel + b\varrho = 0$, the time evolution of $B_\parallel$ provides information on the first-order pressure $\mathbb{p} = b\varrho$ or on the first-order density ϱ. Now, to exhibit the equation satisfied by $B_\parallel$, we pass to the limit in (67). Let us start with the initial condition term. From the assumptions and the discussion about the properties of sequences of initial conditions in Section 2.3, we have $\varrho_0^\varepsilon \rightharpoonup \varrho_0$ in $L^\kappa(\mathbb{T}^3)-$weak with $\kappa = \min\{2, \gamma\}$, $\overline{\rho^\varepsilon} \to 1$ and $B_{0\parallel}^\varepsilon \rightharpoonup B_{0\parallel}$ in $L^2(\mathbb{T}^3)-$weak. It follows that $\mathbb{B}_{0\parallel}^\varepsilon \rightharpoonup \mathbb{B}_{0\parallel} := (B_{0\parallel} - \varrho_0)/\mathbb{c}$ in $\mathcal{D}'(\mathbb{T}^3)$. Observe that $\mathbb{B}_{0\parallel} = B_{0\parallel}$ if and only if $B_{0\parallel} + b\varrho_0 = 0$. In view of (6), the two conditions $\nabla_\perp \cdot v_{0\perp} = 0$ and $B_{0\parallel} + b\varrho_0 = 0$ are related to a preparation of the initial data to avoid fast time oscillations. Still, in our framework, we can incorporate general data satisfying $\mathbb{B}_{0\parallel} \not\equiv B_{0\parallel}$ (and also $\nabla_\perp \cdot v_{0\perp} \not\equiv 0$, see Section 4.5).

It is obvious that $\partial_t \mathbb{B}_\parallel^\varepsilon \rightharpoonup \partial_t \mathbb{B}_\parallel = \partial_t B_\parallel$ in $\mathcal{D}'(\mathbb{R}_+^* \times \mathbb{T}^3)$. We next deal with the linear terms of (67). Using weak convergence of $B_\parallel^\varepsilon$, yielded by the first statement of Lemma 2, we can pass to the limit, in the distributional sense, in all linear diffusive terms of (67). Using the fifth statement of Lemma 3, we obtain $\partial_\parallel(\rho^\varepsilon v_\parallel^\varepsilon) \rightharpoonup \partial_\parallel v_\parallel$ in $\mathcal{D}'(\mathbb{R}_+^* \times \mathbb{T}^3)$. Using weak convergence of $v_\parallel^\varepsilon$ and the strong convergence of $B_\perp^\varepsilon$, given by respectively the first statement of Lemma 3 and the third statement of Lemma 2, we obtain $\nabla_\perp \cdot (v_\parallel^\varepsilon B_\perp^\varepsilon) \rightharpoonup \nabla_\perp \cdot (v_\parallel B_\perp)$ in $\mathcal{D}'(\mathbb{R}_+^* \times \mathbb{T}^3)$. Finally it remains to pass in the limit in the term $\nabla_\perp \cdot (\mathbb{B}_\parallel^\varepsilon v_\perp^\varepsilon)$. On the one hand, we have weak compactness for $\mathbb{B}_\parallel^\varepsilon$, and on the other hand, we have only weak compactness for $v_\perp^\varepsilon$, because, despite the strong compactness of solenoidal part $\mathbb{P}_\perp v_\perp^\varepsilon$, the irrotational part $\mathbb{Q}_\perp v_\perp^\varepsilon$ converges only weakly (see Lemma 3). Therefore, to pass to the limit in this term, we will use Lemma 14 of Appendix A, for which we show below that its hypotheses, splitted in three points, are satisfied. 1) We first recall that $\mathbb{c}\mathbb{B}_\parallel^\varepsilon = B_\parallel^\varepsilon - (\varrho^\varepsilon/\overline{\rho^\varepsilon})$. Using the second statement of Lemma 1 together with the first statement of Lemma 2, for $\kappa = \min\{2, \gamma\}$, we obtain that $\mathbb{B}_\parallel^\varepsilon \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+; (L^\kappa + L^2)(\mathbb{T}^3)) \hookrightarrow L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^\kappa(\mathbb{T}^3))$. Therefore, by weak compactness, we obtain $\mathbb{c}\mathbb{B}_\parallel^\varepsilon \rightharpoonup \mathbb{c}\mathbb{B}_\parallel = B_\parallel - \varrho$ in $L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^\kappa(\mathbb{T}^3))-$weak$-*$. 2) The uniform bound $v_\perp^\varepsilon \in L^2_{\mathrm{loc}}(\mathbb{R}_+; L^6(\mathbb{T}^3))$ and Lemma 4.3 in [7] imply $\|v_\perp^\varepsilon(t, \cdot + h) - v_\perp^\varepsilon(t, \cdot)\|_{L^2_{\mathrm{loc}}(\mathbb{R}_+; L^6(\mathbb{T}^3))} \to 0$, as $|h| \to 0$, uniformly with respect to ε. 3) Using the uniform $L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^\kappa(\mathbb{T}^3))-$bound for $\mathbb{B}_\parallel^\varepsilon$, the uniform $L^2_{\mathrm{loc}}(\mathbb{R}_+; L^6(\mathbb{T}^3))-$bound for v^ε, the uniform $L^2_{\mathrm{loc}}(\mathbb{R}_+; L^6 \cap H^1(\mathbb{T}^3))-$bound for $B_\perp^\varepsilon$, and the uniform $L^2_{\mathrm{loc}}(\mathbb{R}_+; L^q(\mathbb{T}^3))-$bound for $\rho^\varepsilon v_\parallel^\varepsilon$, with $q = 6\gamma/(6+\gamma)$, we obtain from equation (67) and Hölder inequality

$$
\begin{aligned}
\partial_t \mathbb{B}_\parallel^\varepsilon \in L^2_{\mathrm{loc}}(\mathbb{R}_+; W^{-1,6\kappa/(6+\kappa)}(\mathbb{T}^3)) \; &+ L^1_{\mathrm{loc}}(\mathbb{R}_+; W^{-1,3}(\mathbb{T}^3)) \\
&+ L^2_{\mathrm{loc}}(\mathbb{R}_+; (W^{-1,q} + H^{-1})(\mathbb{T}^3)) \hookrightarrow L^1_{\mathrm{loc}}(\mathbb{R}_+; W^{-1,1}(\mathbb{T}^3)) \, ,
\end{aligned}
$$

where the last continuous injection comes from Sobolev embeddings. Gathering points 1) to 3), we can apply Lemma 14 of Appendix A with $g^\varepsilon = \mathbb{B}_\parallel^\varepsilon$, $h^\varepsilon = v_\perp^\varepsilon$, $p_1 = \infty$, $q_1 = 2$ $(1/p_1 + 1/q_1 < 1)$, $p_2 = \kappa$, and $q_2 = 6$ $(1/p_2 + 1/q_2 \leq 2/3 + 1/6 < 1$, for $\gamma, \kappa > 3/2)$, to obtain $\nabla_\perp \cdot (\mathbb{B}_\parallel^\varepsilon v_\perp^\varepsilon) \rightharpoonup \nabla_\perp \cdot (\mathbb{B}_\parallel v_\perp)$ in $\mathcal{D}'(\mathbb{R}_+^* \times \mathbb{T}^3)$. In conclusion, the weak formulation (67) converges to

$$
\begin{aligned}
&\mathbb{c} \int_\Omega dx \, \mathbb{B}_{0\parallel} \, \psi_\parallel(0) + \mathbb{c} \int_0^\infty dt \int_\Omega dx \, \big(\mathbb{B}_\parallel \, (\partial_t + v_\perp \cdot \nabla_\perp) \psi_\parallel \\
&\qquad - v_\parallel \, B_\perp \cdot \nabla_\perp \psi_\parallel - v_\parallel \, \partial_\parallel \psi_\parallel + \eta_\perp B_\parallel \, \Delta_\perp \psi_\parallel + \eta_\parallel B_\parallel \, \Delta_\parallel \psi_\parallel \big) = 0 \, .
\end{aligned}
$$

Knowing (by passing to the weak limit) that $\nabla_\perp \cdot B_\perp = 0$, we recover the first equation of (16) with the initial data prescribed in (17).

5. Asymptotic analysis in the whole space

This section is devoted to the proof of Theorem 2. As in the periodic case, we first obtain some weak and strong compactness properties for the same sequences. Since in the whole case the density is only locally integrable in space, some of these compactness results need different proofs.

5.1. Compactness of ρ^ε and ϱ^ε

Here, we aim at proving the following lemma.

Lemma 6. *There exists a generic constant $C > 0$, which may depend on C_0, a, and γ such that the sequences ρ^ε and $\varrho^\varepsilon := (\rho^\varepsilon - 1)/\varepsilon$ satisfy the following properties.*

$$\begin{aligned}
&\|\rho^\varepsilon\|_{L^\infty_{\mathrm{loc}}(\mathbb{R}_+;L^\gamma_{\mathrm{loc}}(\mathbb{R}^3))} \le C\,, \quad \text{and} \quad (\rho^\varepsilon - 1) \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^\gamma_2 \cap H^{-\alpha}(\mathbb{R}^3))\,, \quad \forall \gamma > 1\,, \quad \alpha \ge 1/2\,,\\
&\|\rho^\varepsilon - 1\|_{L^\infty_{\mathrm{loc}}(\mathbb{R}_+;L^\gamma(\mathbb{R}^3))} \le C\varepsilon^{2/\gamma}\,, \quad \text{and} \quad \|\rho^\varepsilon - 1\|_{L^\infty_{\mathrm{loc}}(\mathbb{R}_+;L^2(\mathbb{R}^3))} \le C\varepsilon\,, \quad \forall \gamma \ge 2\,,\\
&\|\rho^\varepsilon - 1\|_{L^\infty_{\mathrm{loc}}(\mathbb{R}_+;L^\gamma_{\mathrm{loc}}(\mathbb{R}^3))} \le C\varepsilon\,, \quad \text{for all } \ 1 < \gamma < 2\,,\\
&\rho^\varepsilon \longrightarrow 1 \quad \text{in} \quad L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^\gamma_2 \cap L^\gamma_{\mathrm{loc}}(\mathbb{R}^3))-\text{strong}\,, \quad \forall \gamma > 1\,,\\
&\|\varrho^\varepsilon\|_{L^\infty_{\mathrm{loc}}(\mathbb{R}_+;L^\kappa_2 \cap L^\kappa_{\mathrm{loc}} \cap H^{-\alpha}(\mathbb{R}^3))} \le C\,, \quad \kappa = \min\{2,\gamma\}\,, \quad \alpha \ge 1/2\\
&\varrho^\varepsilon \rightharpoonup \varrho \quad \text{in} \quad L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^\kappa_{\mathrm{loc}} \cap H^{-\alpha}(\mathbb{R}^3))-\text{weak}-*\,, \quad \kappa = \min\{2,\gamma\}\,, \quad \alpha \ge 1/2\,.
\end{aligned}$$

Proof. We start with the first bound of the first assertion of Lemma 6. We first claim that $\rho^\varepsilon \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^\gamma_{\mathrm{loc}}(\mathbb{R}^3))$ if $\rho^\varepsilon \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^1_{\mathrm{loc}}(\mathbb{R}^3))$. Indeed, using the convexity of the power function $\mathbb{R}^+ \ni x \mapsto x^\gamma$ (with $\gamma > 1$), and energy inequality (30)-(32) with the pressure term Π_3, there exists a constant c_0, such that for any compact set $K \subset \mathbb{R}^3$, $0 \le \int_K dx\, \{(\rho^\varepsilon)^\gamma - \gamma\rho^\varepsilon + \gamma - 1\} \le c_0$. Then, $\int_K dx\,(\rho^\varepsilon)^\gamma \le c_0 + (\gamma - 1)|K| + \gamma \int_K dx\, \rho^\varepsilon < \infty$, if $\rho^\varepsilon \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^1_{\mathrm{loc}}(\mathbb{R}^3))$. It remains to prove this last property. For this, we consider a test function $\varphi \in \mathscr{C}^\infty_c(\mathbb{R}^3)$, such that $\varphi \ge 0$, and $\varphi \equiv 1$ on a subset K of its support S. From the mass conservation law and the energy estimate, we obtain

$$\frac{d}{dt}\int_S dx\, \rho^\varepsilon \varphi = \int_S dx\, \rho^\varepsilon v^\varepsilon \cdot \nabla\varphi \le \|2\nabla\sqrt{\varphi}\|_{L^\infty(\mathbb{R}^3)} \Big(\int_{\mathbb{R}^3} dx\, \rho^\varepsilon |v^\varepsilon|^2\Big)\Big(\int_S dx\, \rho^\varepsilon\varphi\Big) \le \mathfrak{c}_0 \Big(\int_S dx\, \rho^\varepsilon\varphi\Big),$$

where the constant $\mathfrak{c}_0$ depends on C_0 and φ. This inequality leads to $\int_K dx\, |\rho^\varepsilon| \le \int_S dx\, \rho^\varepsilon\varphi \le \exp(\mathfrak{c}_0 t) \int_S dx\, \rho^\varepsilon_0\varphi$, which shows that $\rho^\varepsilon \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^1_{\mathrm{loc}}(\mathbb{R}^3))$. It can also be shown from the second and third statements of Lemma 6. Now, from the uniform bound (36) and energy inequality (30)-(32) with the pressure term Π_3 defined by (38), we obtain $\Pi_3(\rho^\varepsilon) \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^1(\mathbb{R}^3))$ uniformly with respect to ε. From this bound and Lemma 12 (with $f = \rho^\varepsilon$ and $\bar f = 1$), we obtain, for any $T \in (0, +\infty)$,

$$\begin{aligned}
\sup_{t\in[0,T]} \int_{\mathbb{R}^3} dx\, \{|\rho^\varepsilon - 1|^2 \mathbb{1}_{\{|\rho^\varepsilon - 1| \le \delta\}} + |\rho^\varepsilon - 1|^\gamma \mathbb{1}_{\{|\rho^\varepsilon - 1| > \delta\}}\} &= \sup_{t\in[0,T]} \int_{\mathbb{R}^3} dx\, \mathfrak{Z}^{\gamma,1}_{2,\delta}(\rho^\varepsilon)\\
\le \sup_{t\in[0,T]} \frac{1}{\kappa_1}\int_{\mathbb{R}^3} dx\, \Pi_{1,\gamma}(\rho^\varepsilon) = \sup_{t\in[0,T]} \frac{(\gamma-1)\varepsilon^2}{\kappa_1 a} \int_{\mathbb{R}^3} dx\, \Pi_3(\rho^\varepsilon) &\le \frac{C_0(\gamma-1)\varepsilon^2}{\kappa_1 a}\,. \qquad (68)
\end{aligned}$$

This inequality implies the bound in $L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^\gamma_2(\mathbb{R}^3))$ in the second part of the first statement of Lemma 6, and also the strong convergence of ρ^ε to 1 in $L^\infty_{\mathrm{loc}}(\mathbb{R}_+, L^\gamma_2(\mathbb{R}^3))$ in the fourth statement of Lemma 6. To complete the proof of the second part of the first statement of Lemma 6, we have to show that $L^\gamma_2(\mathbb{R}^3) \hookrightarrow H^{-\alpha}(\mathbb{R}^3)$, with $\alpha \ge 1/2$. This embedding is obvious for $\gamma = 2$. For $\gamma \neq 2$, from the definition of the space $L^\gamma_2(\mathbb{R}^3)$, the density $\rho^\varepsilon(t) - 1$ can be splitted into parts $d_1(t)^\varepsilon := (\rho^\varepsilon(t) - 1)\mathbb{1}_{\{|\rho^\varepsilon(t)-1| \le \delta\}} \in L^2(\mathbb{R}^3)$ and $d^\varepsilon_2(t) := (\rho^\varepsilon(t) - 1)\mathbb{1}_{\{|\rho^\varepsilon(t)-1| > \delta\}} \in L^\gamma(\mathbb{R}^3)$. The part $d^\varepsilon_1(t)$ is obviously in $H^{-\alpha}(\mathbb{R}^3)$ with $\alpha \ge 1/2$. For the part $d^\varepsilon_2(t)$ we proceed as follows. By Sobolev embeddings, we have $H^\alpha(\mathbb{R}^3) \hookrightarrow L^{\gamma'}(\mathbb{R}^3)$ with $1/\gamma + 1/\gamma' = 1$, and $\alpha > 1/2$ since $\gamma > 3/2$. By duality we then obtain $d^\varepsilon_2(t) \in L^\gamma(\mathbb{R}^3) \hookrightarrow H^{-\alpha}(\mathbb{R}^3)$. Therefore $(\rho^\varepsilon(t) - 1) \in H^{-\alpha}(\mathbb{R}^3)$, with $\alpha \ge 1/2$. We continue with the second assertion of Lemma 6. Estimate (68) and the inequality $|\rho^\varepsilon - 1|^2 \ge |\rho^\varepsilon - 1|^\gamma$, for $\gamma \ge 2$ and $|\rho^\varepsilon - 1| \le \delta < 1$, imply the first part of the second statement of Lemma 6. The second part of this second statement appeals to Lemma 11. Indeed, from this lemma, for $\gamma \ge 2$, we obtain $|\rho^\varepsilon - 1|^2 \le \varepsilon^2(\gamma-1)/(\nu_1 a)\Pi_3(\rho^\varepsilon)$, which gives $\|\rho^\varepsilon - 1\|_{L^\infty_{\mathrm{loc}}(\mathbb{R}_+;L^2(\mathbb{R}^3))} \le \varepsilon\sqrt{C_0(\gamma-1)/(\nu_1 a)}$. We continue with the third assertion of Lemma 6. Using Cauchy–Schwarz inequality and estimate (68), we obtain, for any compact set $K \subset \mathbb{R}^3$, and any $T \in (0, +\infty)$,

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$$\sup_{t\in[0,T]} \|\rho^\varepsilon(t)-1\|^\gamma_{L^\gamma(K)} \le |K|^{1-\gamma/2} \sup_{t\in[0,T]} \Big(\int_K dx\, |\rho^\varepsilon-1|^2 \mathbb{1}_{\{|\rho^\varepsilon-1|\le\delta\}}\Big)^{\gamma/2}$$
$$+ \sup_{t\in[0,T]} \int_K dx\, |\rho^\varepsilon-1|^\gamma \mathbb{1}_{\{|\rho^\varepsilon-1|>\delta\}}$$
$$\le C(|K|, C_0, a,\gamma)(\varepsilon^\gamma+\varepsilon^2)\,,$$

which justifies the third assertion of Lemma 6. Then, the strong convergence of ρ^ε to 1 in $L^\infty_{\rm loc}(\mathbb{R}_+, L^\gamma_{\rm loc}(\mathbb{R}^3))$ in the fourth statement of Lemma 6, is obtained from the first part of the second statement of Lemma 6, and the third statement of Lemma 6. We continue with the fifth statement of Lemma 6. The uniform bound in $L^\infty_{\rm loc}(\mathbb{R}_+; L^\kappa_{\rm loc}(\mathbb{R}^3))$ for ϱ^ε comes from the second part of the second statement of Lemma 6, and the third statement of Lemma 6. For the uniform bound of ϱ^ε in $L^\infty_{\rm loc}(\mathbb{R}_+; L^\kappa_2(\mathbb{R}^3))$, we distinguish two cases according to the value of γ. For $\gamma\ge 2$, since $\kappa=2$, the second part of the second statement of Lemma 6 implies the fifth one. For $1<\gamma<2$, estimate (68) leads to

$$\sup_{t\in[0,T]} \int_{\mathbb{R}^3} dx \,\Big\{ \Big|\frac{\rho^\varepsilon-1}{\varepsilon}\Big|^2 \mathbb{1}_{\{|\frac{\rho^\varepsilon-1}{\varepsilon}|\le\frac{\delta}{\varepsilon}\}} + \varepsilon^{\gamma-2}\Big|\frac{\rho^\varepsilon-1}{\varepsilon}\Big|^\gamma \mathbb{1}_{\{|\frac{\rho^\varepsilon-1}{\varepsilon}|>\frac{\delta}{\varepsilon}\}}\Big\} \le \frac{C_0(\gamma-1)}{\kappa_1 a}\,,$$

for any $T\in(0,+\infty)$. This last estimate and inequality $\varepsilon^{\gamma-2}>1$, imply the bound of ϱ^ε in $L^\infty_{\rm loc}(\mathbb{R}_+; L^\kappa_2(\mathbb{R}^3))$. To complete the fifth assertion of Lemma 6, we observe, as above, that we have the embedding $L^\kappa_2(\mathbb{R}^3)\hookrightarrow H^{-\alpha}(\mathbb{R}^3)$, with $\alpha\ge 1/2$ and $\kappa=\min\{2,\gamma\}$. Finally, the sixth statement of Lemma 6 is obtained from the fifth assertion and weak compactness properties. This ends the proof of Lemma 6. □

5.2. Compactness of B^ε

Here, we aim at proving the following lemma.

Lemma 7. *The sequence B^ε satisfies the following properties.*

$$B^\varepsilon \rightharpoonup B \quad \text{in} \quad L^2_{\rm loc}(\mathbb{R}_+; L^6\cap H^1(\mathbb{R}^3)) - \text{weak} \cap L^\infty_{\rm loc}(\mathbb{R}_+; L^2(\mathbb{R}^3)) - \text{weak}-*\,,$$
$$\nabla_\varepsilon\cdot B^\varepsilon \rightharpoonup \nabla_\perp\cdot B_\perp = 0 \quad \text{in} \quad L^2_{\rm loc}(\mathbb{R}_+; L^2(\mathbb{R}^3)) - \text{weak}\,,$$
$$B^\varepsilon_\perp \longrightarrow B_\perp \quad \text{in} \quad L^r_{\rm loc}(\mathbb{R}_+; L^2_{\rm loc}(\mathbb{R}^3)) - \text{strong}\,, \quad 1\le r<\infty\,.$$

Proof. The proof of Lemma 7 is similar to the proof of Lemma 2. □

5.3. Compactness of v^ε and $\rho^\varepsilon v^\varepsilon$

Here, we aim at proving the following Lemma.

Lemma 8. *Assume $\gamma>3/2$. Let $\mathfrak{s} := \max\{1/2, 3/\gamma-1\}\in[1/2,1)$. The sequences v^ε and $\rho^\varepsilon v^\varepsilon$ satisfy the following properties.*

$$v^\varepsilon \rightharpoonup v \quad \text{in} \quad L^2_{\rm loc}(\mathbb{R}_+; L^6\cap H^1(\mathbb{R}^3)) - \text{weak}\,,$$
$$\nabla_\varepsilon\cdot v^\varepsilon \rightharpoonup \nabla_\perp\cdot v_\perp = 0 \quad \text{in} \quad L^2_{\rm loc}(\mathbb{R}_+; L^2(\mathbb{R}^3)) - \text{weak}\,,$$
$$\mathbb{Q}_\perp v^\varepsilon_\perp \rightharpoonup 0 \quad \text{in} \quad L^2_{\rm loc}(\mathbb{R}_+; L^6\cap H^1(\mathbb{R}^3)) - \text{weak}\,,$$
$$\mathbb{P}_\perp v^\varepsilon_\perp \longrightarrow \mathbb{P}_\perp v_\perp = v_\perp \quad \text{in} \quad L^2_{\rm loc}(\mathbb{R}_+; L^p_{\rm loc}\cap H^s_{\rm loc}(\mathbb{R}^3)) - \text{strong}\,, \quad 1\le p<6\,, \quad 0\le s<1\,,$$

$$\rho^\varepsilon v^\varepsilon \longrightarrow v \quad \text{in} \quad L^2_{\text{loc}}(\mathbb{R}_+; L^q_{\text{loc}} \cap H^{-\sigma}(\mathbb{R}^3))-\text{weak}\,, \quad \forall\, \sigma \geq \mathfrak{s}\,, \quad q = 6\gamma/(6+\gamma)\,,$$
$$\rho^\varepsilon v^\varepsilon - v^\varepsilon \longrightarrow 0 \quad \text{in} \quad L^2_{\text{loc}}(\mathbb{R}_+; L^q_{\text{loc}}(\mathbb{R}^3))-\text{strong}\,, \quad q = 6\gamma/(6+\gamma)\,,$$
$$\mathbb{P}_\perp(\rho^\varepsilon v^\varepsilon_\perp) \longrightarrow \mathbb{P}_\perp v_\perp = v_\perp \quad \text{in} \quad L^2_{\text{loc}}(\mathbb{R}_+; L^q_{\text{loc}}(\mathbb{R}^3))-\text{strong}\,, \quad q = 6\gamma/(6+\gamma)\,.$$

Proof. We start with the first statement of Lemma 8. The proof of the first statement of Lemma 8 is similar to the one of the first statement of Lemma 3 except for the bound $L^2_{\text{loc}}(\mathbb{R}_+; L^2(\mathbb{R}^3))$. We consider the decomposition $v^\varepsilon = v^\varepsilon_1 + v^\varepsilon_2$, where $v^\varepsilon_1 := v^\varepsilon \mathbb{1}_{\{|\rho^\varepsilon - 1| \leq \delta\}}$, and $v^\varepsilon_2 := v^\varepsilon \mathbb{1}_{\{|\rho^\varepsilon - 1| > \delta\}}$. For v^ε_1, since $|\rho^\varepsilon - 1| \leq \delta$, we obtain $1 - \delta \leq |\rho^\varepsilon|$ and thus

$$\|v^\varepsilon_1\|_{L^\infty_{\text{loc}}(\mathbb{R}^+; L^2(\mathbb{R}^3))} \leq (1-\delta)^{-1/2} \|\rho^\varepsilon |v^\varepsilon|^2\|^{1/2}_{L^\infty_{\text{loc}}(\mathbb{R}^+; L^1(\mathbb{R}^3))} \leq \sqrt{C_0/(1-\delta)}\,, \tag{69}$$

where we have used energy inequality (30)-(32) with the pressure term Π_3 defined by (38). For v^ε_2, using in order, Hölder inequality ($1/\gamma + 1/\gamma' = 1$), estimate (68), Gagliardo–Nirenberg interpolation inequality ($1/(2\gamma') = \theta/2 + (1-\theta)(1/2 - 1/3)$, i.e., $\theta = 1 - 3/(2\gamma) \in (0,1)$, since $3/2 < \gamma < \infty$), and Young inequality ($ab \leq \theta a^{1/\theta} + (1-\theta) b^{1/(1-\theta)}$), we obtain

$$\begin{aligned}
\int_{\mathbb{R}^3} dx\, |v^\varepsilon_2|^2 &\leq \delta^{-1} \int_{\mathbb{R}^3} dx\, |v^\varepsilon|^2 |\rho^\varepsilon - 1| \mathbb{1}_{\{|\rho^\varepsilon - 1| > \delta\}} \\
&\leq \delta^{-1} \|(\rho^\varepsilon - 1)\mathbb{1}_{\{|\rho^\varepsilon - 1| > \delta\}}\|_{L^\gamma(\mathbb{R}^3)} \|v^\varepsilon\|^2_{L^{2\gamma'}(\mathbb{R}^3)} \\
&\leq \delta^{-1} \left(\frac{C_0(\gamma-1)}{\kappa_1 a}\right)^{1/\gamma} \varepsilon^{2/\gamma} \|v^\varepsilon\|^2_{L^{2\gamma'}(\mathbb{R}^3)} \\
&\leq \delta^{-1} \left(\frac{C_0(\gamma-1)}{\kappa_1 a}\right)^{1/\gamma} \varepsilon^{2/\gamma} \|v^\varepsilon\|^{2\theta}_{L^2(\mathbb{R}^3)} \|\nabla v^\varepsilon\|^{2(1-\theta)}_{L^2(\mathbb{R}^3)} \\
&\leq \delta^{-1} \left(\frac{C_0(\gamma-1)}{\kappa_1 a}\right)^{1/\gamma} \varepsilon^{2/\gamma} \left(\theta \|v^\varepsilon\|^2_{L^2(\mathbb{R}^3)} + (1-\theta)\|\nabla v^\varepsilon\|^2_{L^2(\mathbb{R}^3)}\right).
\end{aligned} \tag{70}$$

From energy inequality (30)-(32) with the pressure term Π_3, we obtain $\nabla v^\varepsilon \in L^2_{\text{loc}}(\mathbb{R}_+; L^2(\mathbb{R}^3))$, uniformly with respect to ε. Using this last bound and (69), a local time integration of estimate (70) implies that there exist two constants K_0 and K_1 (depending on C_0, γ, κ_1, a and δ) such that

$$\|v^\varepsilon\|^2_{L^2_{\text{loc}}(\mathbb{R}^+; L^2(\mathbb{R}^3))} \leq K_0 + K_1 \varepsilon^{2/\gamma} \|v^\varepsilon\|^2_{L^2_{\text{loc}}(\mathbb{R}^+; L^2(\mathbb{R}^3))}\,. \tag{71}$$

For ε small enough, this last inequality implies $v^\varepsilon \in L^2_{\text{loc}}(\mathbb{R}_+; L^2(\mathbb{R}^3))$ and thus $v^\varepsilon \in L^2_{\text{loc}}(\mathbb{R}_+; L^6 \cap H^1(\mathbb{R}^3))$. We note that estimates (70)-(71) also imply $\varepsilon^{-\eta} v^\varepsilon_2 \in L^2_{\text{loc}}(\mathbb{R}_+; L^2(\mathbb{R}^3))$, uniformly with respect to ε, for $0 \leq \eta \leq 2/3$ (since $\gamma > 3/2$). We continue with the other statements of Lemma 8. Using Lemma 6 and energy inequality (30)-(32) with the pressure term Π_3 defined by (38), the proof of statements two to seven of Lemma 8 is similar to the proof of their counterparts of Lemma 3 for periodic domains. □

We continue with an auxiliary lemma, which will be useful to pass to the limit in the term $\rho^\varepsilon v^\varepsilon_\perp \otimes v^\varepsilon_\perp$. Contrary to the periodic case, we are not able to get the strong convergence (50) for U^ε. Indeed, we can still prove locally strong compactness in space-time for $\mathcal{U}^\varepsilon$ (using an Aubin–Lions theorem and estimates (75)-(76) below), but returning to U^ε, via the group of isometry $\mathcal{S}$, such local strong compactness seems unaccessible since $\mathcal{S}$ is non local. To recover some strong compactness for U^ε we consider a truncated version of U^ε and use the uniform integrability in space of such truncated sequences as well as the energy inequality and the strong convergence of ρ^ε.

Lemma 9. *Assume $\gamma > 3/2$. Let us define $U^\varepsilon := {}^t(\phi^\varepsilon, {}^t\Phi^\varepsilon)$, where $\phi^\varepsilon := b\varrho^\varepsilon + B^\varepsilon_\parallel$, and $\Phi^\varepsilon := \mathbb{Q}_\perp(\rho^\varepsilon v^\varepsilon_\perp)$, with $\varrho^\varepsilon := (\rho^\varepsilon - 1)/\varepsilon$. We also define $\mathcal{U}^\varepsilon := \mathcal{S}(-t/\varepsilon)U^\varepsilon$, where the group of isometry $\{\mathcal{S}(\tau)\,;\tau \in \mathbb{R}\}$ is the same as the one defined in Section 4.5 with now $\Omega = \mathbb{R}^3$, except that we substitute c for c^ε. Finally, we recall that $\kappa = \min\{2, \gamma\}$, and we set $\mathsf{k} = \max\{1/2, 3/(2\gamma)\} \in [1/2, 1)$ and $\mathfrak{s} := \max\{1/2, 3/\gamma - 1\} \in [1/2, 1)$. Then,*

1. *We have the following uniform (with respect to ε) bounds*

$$\phi^\varepsilon \in L^\infty_{\rm loc}(\mathbb{R}_+; H^{-\alpha}(\mathbb{R}^3))\,, \quad \alpha \geq \tfrac{1}{2}\,, \tag{72}$$

$$\rho^\varepsilon v^\varepsilon \in L^\infty_{\rm loc}\big(\mathbb{R}_+; \big(L^2 + L^{2\gamma/(\gamma+1)}\big) \cap L^{2\gamma/(\gamma+1)}_{\rm loc} \cap H^{-\beta}(\mathbb{R}^3)\big)\,, \quad \beta \geq \tfrac{3}{2\gamma}\,. \tag{73}$$

$$\varrho^\varepsilon v^\varepsilon \in L^2_{\rm loc}\big(\mathbb{R}_+; \big(L^{3/2} + L^{6\kappa/(6+\kappa)}\big) \cap L^{6\kappa/(6+\kappa)}_{\rm loc} \cap H^{-\varkappa}(\mathbb{R}^3)\big)\,, \quad \varkappa \geq \mathfrak{s}\,. \tag{74}$$

2. *There exists a constant C, independent of ε, such that*

$$\|\mathcal{U}^\varepsilon\|_{L^\infty_{\rm loc}(\mathbb{R}_+;H^{-\sigma}(\mathbb{R}^3))} \leq C\,, \quad \sigma \geq \mathsf{k}\,, \tag{75}$$

$$\|\partial_t \mathcal{U}^\varepsilon\|_{L^2_{\rm loc}(\mathbb{R}_+;H^{-r}(\mathbb{R}^3))} \leq C\,, \quad r > \left(\tfrac{5}{2}\right)^+\,, \tag{76}$$

3. *There exists a function $\widetilde{\mathcal{U}}^\varepsilon := {}^t(\widetilde{\psi}^\varepsilon, {}^t\widetilde{\Psi}^\varepsilon) \in L^\infty_{\rm loc}(\mathbb{R}_+; L^2(\mathbb{R}^3;\mathbb{R}^3))$, a constant C, independent of ε, and a function $\omega : \mathbb{R}_+ \mapsto \mathbb{R}_+$, continuous in the neighborhood of zero, with $\omega(\varepsilon) \to 0$, as $\varepsilon \to 0_+$, such that, for $\sigma \geq \mathsf{k}$,*

$$\|\widetilde{\mathcal{U}}^\varepsilon\|_{L^\infty_{\rm loc}(\mathbb{R}_+;L^2(\mathbb{R}^3)} \leq C\,, \tag{77}$$

$$\|\mathcal{U}^\varepsilon - \widetilde{\mathcal{U}}^\varepsilon\|_{L^\infty_{\rm loc}(\mathbb{R}_+;H^{-\sigma}(\mathbb{R}^3))} \lesssim \omega(\varepsilon)\,, \tag{78}$$

$$\|U^\varepsilon - \mathcal{S}(t/\varepsilon)\widetilde{\mathcal{U}}^\varepsilon\|_{L^\infty_{\rm loc}(\mathbb{R}_+;H^{-\sigma}(\mathbb{R}^3))} \lesssim \omega(\varepsilon)\,, \tag{79}$$

$$\|\mathbb{Q}_\perp v^\varepsilon_\perp - \mathcal{S}_2(t/\varepsilon)\widetilde{\mathcal{U}}^\varepsilon\|_{L^2_{\rm loc}(\mathbb{R}_+;H^{-\sigma}(\mathbb{R}^3))} \lesssim \omega(\varepsilon) + \varepsilon\,. \tag{80}$$

Proof. We start with the point 1 of Lemma 9, beginning with (72). Using the fifth statement of Lemma 6 for an estimate of ϱ^ε, the first statement of Lemma 7 for an estimate of $B^\varepsilon_\parallel$, the embedding $(L^\kappa + L^2)(\mathbb{R}^3) \hookrightarrow H^{-\alpha}$, with $\alpha \geq 1/2$ and $\kappa > 3/2$, and the definition $\phi^\varepsilon := b\varrho^\varepsilon + B^\varepsilon_\parallel$, we obtain (72). We continue with (73), by recasting $\rho^\varepsilon v^\varepsilon$ as

$$\rho^\varepsilon v^\varepsilon = (\sqrt{\rho^\varepsilon} v^\varepsilon)\sqrt{\rho^\varepsilon}\mathbb{1}_{\{|\rho^\varepsilon-1|\leq\delta\}} + (\sqrt{\rho^\varepsilon} v^\varepsilon)\frac{\sqrt{\rho^\varepsilon}}{\sqrt{|\rho^\varepsilon - 1|}}\sqrt{|\rho^\varepsilon - 1|}\mathbb{1}_{\{|\rho^\varepsilon-1|>\delta\}}. \tag{81}$$

The first term of the right-hand side of (81) is the product of the function $\sqrt{\rho^\varepsilon} v^\varepsilon \in L^\infty_{\rm loc}(\mathbb{R}_+; L^2(\mathbb{R}^3))$ and the function $\rho^\varepsilon \mathbb{1}_{\{|\rho^\varepsilon-1|\leq\delta\}} \in L^\infty_{\rm loc}(\mathbb{R}_+; L^\infty(\mathbb{R}^3))$, thus using Hölder inequality this product is in $L^\infty_{\rm loc}(\mathbb{R}_+; L^2(\mathbb{R}^3))$. The second term of the right-hand side of (81) is the triple product of the function $\sqrt{\rho^\varepsilon} v^\varepsilon \in L^\infty_{\rm loc}(\mathbb{R}_+; L^2(\mathbb{R}^3))$, the function $\sqrt{\rho^\varepsilon}/\sqrt{|\rho^\varepsilon - 1|}\mathbb{1}_{\{|\rho^\varepsilon-1|>\delta\}} \in L^\infty_{\rm loc}(\mathbb{R}_+; L^\infty(\mathbb{R}^3))$, and the function $\sqrt{|\rho^\varepsilon - 1|}\mathbb{1}_{\{|\rho^\varepsilon-1|>\delta\}} \in L^\infty_{\rm loc}(\mathbb{R}_+; L^{2\gamma}(\mathbb{R}^3))$, thus, using Hölder inequality, this triple product is in $L^\infty_{\rm loc}(\mathbb{R}_+; L^{2\gamma/(\gamma+1)}(\mathbb{R}^3))$. Using the Sobolev embeddings $\dot{H}^\beta(\mathbb{R}^3) \hookrightarrow \{L^{2\gamma/(\gamma-1)}(\mathbb{R}^3)\,,\ L^2(\mathbb{R}^3)\}$ (with $\gamma > 3/2$), for $\beta \geq 3/(2\gamma)$, by duality we obtain $\{L^2(\mathbb{R}^3)\,, L^{2\gamma/(\gamma+1)}(\mathbb{R}^3)\} \hookrightarrow H^{-\beta}(\mathbb{R}^3)$; hence (73). We continue with the proof of (74). We first consider the decomposition $\varrho^\varepsilon v^\varepsilon = \varrho^\varepsilon v^\varepsilon_1 + \varrho^\varepsilon v^\varepsilon_2$, with v^ε_1 and v^ε_2 defined as in the proof of Lemma 8. From $\varrho^\varepsilon \mathbb{1}_{\{|\rho^\varepsilon-1|\leq\delta\}} \in L^\infty_{\rm loc}(\mathbb{R}_+; L^2(\mathbb{R}^3))$ (fifth statement of Lemma 6) and $v^\varepsilon \in L^2_{\rm loc}(\mathbb{R}_+; L^6(\mathbb{R}^3))$, using Hölder inequality, we obtain $\varrho^\varepsilon v^\varepsilon_1 \in L^2_{\rm loc}(\mathbb{R}_+; L^{3/2}(\mathbb{R}^3))$. From $\varrho^\varepsilon \mathbb{1}_{\{|\rho^\varepsilon-1|>\delta\}} \in L^\infty_{\rm loc}(\mathbb{R}_+; L^\kappa(\mathbb{R}^3))$ (fifth statement of Lemma 6) and $v^\varepsilon \in L^2_{\rm loc}(\mathbb{R}_+; L^6(\mathbb{R}^3))$, using Hölder inequality, we obtain $\varrho^\varepsilon v^\varepsilon_2 \in L^2_{\rm loc}(\mathbb{R}_+; L^{\mathfrak{q}}(\mathbb{R}^3))$, with $\mathfrak{q} = 6\kappa/(6+\kappa)$. Moreover, using the first statement of Lemma 6 and Hölder inequality, we obtain $\varrho^\varepsilon v^\varepsilon \in L^2_{\mathrm{loc}}(\mathbb{R}_+; L^{\mathfrak{q}}_{\mathrm{loc}}(\mathbb{R}^3))$. It remains to show the $H^{-\varkappa}$-bound in (74). Since $\varrho^\varepsilon v_1^\varepsilon \in L^2_{\mathrm{loc}}(\mathbb{R}_+; L^{3/2}(\mathbb{R}^3))$, using the Sobolev embedding $L^{3/2}(\mathbb{R}^3) \hookrightarrow H^{-\alpha}(\mathbb{R}^3)$, with $\alpha \geq 1/2$, we obtain $\varrho^\varepsilon v_1^\varepsilon \in L^2_{\mathrm{loc}}(\mathbb{R}_+; H^{-\alpha}(\mathbb{R}^3))$. Since $\varrho^\varepsilon v_2^\varepsilon \in L^2_{\mathrm{loc}}(\mathbb{R}_+; L^{\mathfrak{q}}(\mathbb{R}^3))$, using the Sobolev embedding $L^{\mathfrak{q}}(\mathbb{R}^3) \hookrightarrow H^{-\tilde{\mathfrak{s}}}$ with $\tilde{\mathfrak{s}} \geq (3/\kappa) - 1$ while $(3/\kappa) - 1 \leq \mathfrak{s}$, we get $\varrho^\varepsilon v_2^\varepsilon \in L^2_{\mathrm{loc}}(\mathbb{R}_+; H^{-\tilde{\mathfrak{s}}}(\mathbb{R}^3))$. Combining this two last results, we obtain $\varrho^\varepsilon v^\varepsilon \in L^2_{\mathrm{loc}}(\mathbb{R}_+; H^{-\varkappa}(\mathbb{R}^3))$, with $\varkappa \geq \mathfrak{s}$; hence (74).

We continue with the point 2 of Lemma 9, starting with (75). Using (72)-(73) and since the group $\mathcal{S}$ (resp. the operator $\mathbb{Q}_\perp$) is an isometry (resp. continuous) in $H^\alpha(\mathbb{R}^3)$ with $\alpha \in \mathbb{R}$, we obtain (75). For (76), observe that $\partial_t \mathcal{U}^\varepsilon = \mathcal{S}(-t/\varepsilon) F^\varepsilon$, with F^ε defined by

$$F^\varepsilon := \begin{pmatrix} F_1^\varepsilon \\ F_2^\varepsilon \end{pmatrix} = \begin{cases} F_1^\varepsilon = -\, b\partial_\parallel(\rho^\varepsilon v_\parallel^\varepsilon) + \nabla_\perp \cdot \mathbb{Q}_\perp(\varrho^\varepsilon v_\perp^\varepsilon) - B_\parallel^\varepsilon \nabla_\varepsilon \cdot v^\varepsilon \\ \qquad - (v^\varepsilon \cdot \nabla_\varepsilon) B_\parallel^\varepsilon + (B^\varepsilon \cdot \nabla_\varepsilon) v_\parallel^\varepsilon + (\eta_\perp^\varepsilon \Delta_\perp + \eta_\parallel^\varepsilon \Delta_\parallel) B_\parallel^\varepsilon \,, \\ F_2^\varepsilon = -\, \mathbb{Q}_\perp \nabla_\varepsilon \cdot (\rho^\varepsilon v_\perp^\varepsilon \otimes v^\varepsilon) - (\gamma - 1)\nabla_\perp \Pi_3(\rho^\varepsilon) - \frac{1}{2}\nabla_\perp(|B^\varepsilon|^2) + \partial_\parallel \mathbb{Q}_\perp B_\perp^\varepsilon \\ \qquad + \mathbb{Q}_\perp \nabla_\varepsilon \cdot (B_\perp^\varepsilon \otimes B^\varepsilon) + \mu_\perp^\varepsilon \nabla_\perp(\nabla_\perp \cdot v_\perp^\varepsilon) + \mu_\parallel^\varepsilon \Delta_\parallel \mathbb{Q}_\perp v_\perp^\varepsilon + \lambda^\varepsilon \nabla_\perp(\nabla_\varepsilon \cdot v^\varepsilon) \,. \end{cases} \tag{82}$$

Using estimates of point 1 in Lemma 9 and following Step 1 in the proof Lemma 4, we obtain from (82) and $\kappa > \mathfrak{s}$ ($\gamma > 3/2$), $F_1^\varepsilon \in L^2_{\mathrm{loc}}(\mathbb{R}_+; (H^{-\kappa-1} + W^{-1,3/2} + H^{-1})(\mathbb{R}^3)) \hookrightarrow L^2_{\mathrm{loc}}(\mathbb{R}_+; H^{-\kappa-1}(\mathbb{R}^3))$, and $F_2^\varepsilon \in L^2_{\mathrm{loc}}(\mathbb{R}_+; (W^{-1-\delta,1} + W^{-1,1} + H^{-1} + L^2)(\mathbb{R}^3)) \hookrightarrow L^2_{\mathrm{loc}}(\mathbb{R}_+; H^{-r}(\mathbb{R}^3))$, with $r > 5/2 + \delta$, for any $\delta > 0$. Therefore, using the isometry $\mathcal{S}$ in $H^\alpha(\mathbb{R}^3)$ for any $\alpha \in \mathbb{R}$, we obtain (76).

We continue with the point 3 of Lemma 9, starting with (77). For any $\delta > 0$, we define $\widetilde{\psi}^\varepsilon := \mathcal{S}_1(-t/\varepsilon)[\phi^\varepsilon \mathbb{1}_{\{\rho^\varepsilon \leq 1+\delta\}}]$ and $\widetilde{\Psi}^\varepsilon := \mathcal{S}_2(-t/\varepsilon)\mathbb{Q}_\perp(\rho^\varepsilon v_\perp^\varepsilon \mathbb{1}_{\{\rho^\varepsilon \leq 1+\delta\}})$. Clearly, from the uniform bounds above and the isometry $\mathcal{S}$ in $H^\alpha(\mathbb{R}^3)$ for any $\alpha \in \mathbb{R}$, we obtain $\widetilde{\psi}^\varepsilon$, $\widetilde{\Psi}^\varepsilon \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^2(\mathbb{R}^3))$. We continue with the proof of (78). Since $\varrho^\varepsilon \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L_2^\kappa(\mathbb{R}^3))$ and $B_\parallel^\varepsilon \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^2(\mathbb{R}^3))$, using the De la Vallée Poussin criterion, we obtain that ϕ^ε is spatially uniformly integrable in $L^{3/2}(\mathbb{R}^3)$, uniformly in time on any compact time interval. Then, from the fourth statement of Lemma 6, we obtain $\|\phi^\varepsilon \mathbb{1}_{\{\rho^\varepsilon > 1+\delta\}}\|_{L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^{3/2}(\mathbb{R}^3))} \to 0$, as $\varepsilon \to 0$. Therefore, using the isometry $\mathcal{S}$, we obtain the first part of (78), that is $\|\psi^\varepsilon - \widetilde{\psi}^\varepsilon\|_{L^\infty_{\mathrm{loc}}(\mathbb{R}_+; H^{-\sigma}(\mathbb{R}^3))} \to 0$, as $\varepsilon \to 0$. Since $\rho^\varepsilon v_\perp^\varepsilon \mathbb{1}_{\{\rho^\varepsilon > 1+\delta\}} = \sqrt{\rho^\varepsilon} v_\perp^\varepsilon \sqrt{\rho^\varepsilon} \mathbb{1}_{\{\rho^\varepsilon > 1+\delta\}}$, using $\rho^\varepsilon \to 1$ in $L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L_2^\gamma(\mathbb{R}^3))$ − strong (fourth statement of Lemma 6), the uniform bound $\|\rho^\varepsilon |v^\varepsilon|^2\|_{L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^1(\mathbb{R}^3))} \leq C < \infty$, and Hölder inequality, we obtain $\rho^\varepsilon v_\perp^\varepsilon \mathbb{1}_{\{\rho^\varepsilon > 1+\delta\}} \to 0$ in $L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^{2\gamma/(\gamma+1)}(\mathbb{R}^3))$, as $\varepsilon \to 0$. Since the group $\mathcal{S}$ (resp. the operator $\mathbb{Q}_\perp$) is an isometry (resp. continuous) in $H^\alpha(\mathbb{R}^3)$ for any $\alpha \in \mathbb{R}$, using the embedding $L^{2\gamma/(\gamma+1)}(\mathbb{R}^3) \hookrightarrow H^{-\sigma}(\mathbb{R}^3)$ (see above in the proof), we obtain the second part of (78), that is $\|\Psi^\varepsilon - \widetilde{\Psi}^\varepsilon\|_{L^\infty_{\mathrm{loc}}(\mathbb{R}_+; H^{-\sigma}(\mathbb{R}^3))} \to 0$, as $\varepsilon \to 0$. Still using the isometry $\mathcal{S}$, we deduce estimate (79) from (78). It remains to prove (80). Using the continuity of $\mathbb{Q}_\perp$ and (79), we obtain $\|\mathbb{Q}_\perp v_\perp^\varepsilon - \mathcal{S}_2(t/\varepsilon)\widetilde{\mathcal{U}}^\varepsilon\|_{L^2_{\mathrm{loc}}(\mathbb{R}_+; H^{-\sigma}(\mathbb{R}^3))} \leq \|(\rho^\varepsilon - 1) v_\perp^\varepsilon\|_{L^2_{\mathrm{loc}}(\mathbb{R}_+; H^{-\sigma}(\mathbb{R}^3))} + \omega(\varepsilon)$. We split $(\rho^\varepsilon - 1) v_\perp^\varepsilon$ into the part $d_1^\varepsilon := (\rho^\varepsilon - 1) v_\perp^\varepsilon \mathbb{1}_{\{|\rho^\varepsilon - 1| \leq \delta\}}$ and the part $d_2^\varepsilon := (\rho^\varepsilon - 1) v_\perp^\varepsilon \mathbb{1}_{\{|\rho^\varepsilon - 1| > \delta\}}$. Since $L^{3/2}(\mathbb{R}^3) \hookrightarrow H^{-1/2}(\mathbb{R}^3)$, using Hölder inequality and estimate (68), we obtain $\|d_1^\varepsilon\|_{L^2_{\mathrm{loc}}(\mathbb{R}_+; H^{-\sigma}(\mathbb{R}^3))} \leq \|d_1^\varepsilon\|_{L^2_{\mathrm{loc}}(\mathbb{R}_+; H^{-1/2}(\mathbb{R}^3))} \leq \|(\rho^\varepsilon - 1)\mathbb{1}_{\{|\rho^\varepsilon - 1| \leq \delta\}}\|_{L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^2(\mathbb{R}^3))} \|v_\perp^\varepsilon\|_{L^2_{\mathrm{loc}}(\mathbb{R}_+; L^6(\mathbb{R}^3))} \lesssim \varepsilon$. For the part d_2^ε, we distinguish two cases according to the value of γ. For $\gamma \geq 2$, following the same proof as for d_1^ε, we obtain $\|d_2^\varepsilon\|_{L^2_{\mathrm{loc}}(\mathbb{R}_+; H^{-\sigma}(\mathbb{R}^3))} \lesssim \varepsilon$. For $3/2 < \gamma < 2$, we have $\kappa = 3/(2\gamma) > \mathfrak{s} = 3/\gamma - 1$ (since $\gamma > 3/2$). Then, using the Sobolev embeddings $L^{6\gamma/(6+\gamma)}(\mathbb{R}^3) \hookrightarrow H^{-\mathfrak{s}}(\mathbb{R}^3)$, and estimate (68), we obtain $\|d_2^\varepsilon\|_{L^2_{\mathrm{loc}}(\mathbb{R}_+; H^{-\sigma}(\mathbb{R}^3))} \leq \|d_2^\varepsilon\|_{L^2_{\mathrm{loc}}(\mathbb{R}_+; H^{-\kappa}(\mathbb{R}^3))} \leq \|d_2^\varepsilon\|_{L^2_{\mathrm{loc}}(\mathbb{R}_+; H^{-\mathfrak{s}}(\mathbb{R}^3))} \leq \|(\rho^\varepsilon - 1)\mathbb{1}_{\{|\rho^\varepsilon - 1| > \delta\}}\|_{L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^\gamma(\mathbb{R}^3))} \|v_\perp^\varepsilon\|_{L^2_{\mathrm{loc}}(\mathbb{R}_+; L^6(\mathbb{R}^3))} \lesssim \varepsilon^{2/\gamma} \leq \varepsilon$. This ends the proof of Lemma 9. □

5.4. *Passage to the limit in the compressible MHD equations*

Here, we justify the passage to the limit in the weak formulation (18)-(25) of the MHD equations for the whole space. Using Lemmas 7 and 8, the passage to the limit in equation (23) for $B_\perp^\varepsilon$ follows the same proof as the one of the periodic case described in Section 4.4. Using Lemmas 6, 7 and 8, and following the sames lines as the ones of Section 4.5 for the periodic case, we can pass to the limit in almost all terms of

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equation (21) for $\rho^\varepsilon v_\perp^\varepsilon$. Indeed, the only difference with the proof of Section 4.5 is the justification of the limit $\varepsilon \to 0$ for the term $\rho^\varepsilon v_\perp^\varepsilon \otimes v_\perp^\varepsilon$ that we detail below in Lemma 10. In particular, the pressure term $\mathfrak{p}^\varepsilon = p^\varepsilon/\varepsilon^2 + B_\parallel^\varepsilon/\varepsilon + |B^\varepsilon|^2/2$, rewritten as $\mathfrak{p}^\varepsilon = \phi^\varepsilon/\varepsilon + a/\varepsilon^2 + \pi_2^\varepsilon$, with $\pi_2^\varepsilon = (\gamma-1)\Pi_3(\rho^\varepsilon) + |B^\varepsilon|^2/2$, converges weakly to $\pi_1+\pi_2$ in $\mathcal{D}'(\mathbb{R}_+^*\times\mathbb{R}^3)$. More precisely, $\phi^\varepsilon/\varepsilon \rightharpoonup \delta_0(t)\otimes\pi_0 + \pi_1$ in $H^{-1}(\mathbb{R}_+, H^{-r}(\mathbb{R}^3))-$weak, with $r > (5/2)^+$, and $\pi_2^\varepsilon \rightharpoonup \pi_2$ in $L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^1(\mathbb{R}^3))-$weak$-*$, while the constant term ba/ε^2 disappears by spatial integration in (21). As in the periodic case, the term $\delta_0(t)\otimes\pi_0$ cancels the irrotational part $\mathbb{Q}_\perp u_{0\perp}$ of the limit term $u_{0\perp}$, so that the limit initial condition is $\mathbb{P}_\perp u_{0\perp} = \mathbb{P}_\perp v_{0\perp}$. We also obtain $\phi^\varepsilon \rightharpoonup \phi = b\varrho + B_\parallel = 0$ in $H^{-1}(\mathbb{R}_+, H^{-r}(\mathbb{R}^3))-$weak, and the relation $b\varrho + B_\parallel = 0$ holds for a.e. $(t,x) \in]0,+\infty[\times\mathbb{R}^3$, since $b\varrho + B_\parallel \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+; (L^\kappa_{\mathrm{loc}} + L^2)(\mathbb{R}^3))$. Using Lemmas 6, 7 and 8, the passage to limit in equations (22) and (67), for respectively $\rho^\varepsilon v_\parallel^\varepsilon$ and $\mathbb{B}^\varepsilon$, is justified in a similar way as the one described in Sections 4.6 and 4.7 for the periodic case. Finally, to conclude the proof in the case of the whole space, we use the following lemma, which is the counterpart of Lemma 5 with a different proof since we do not have the strong convergence (50).

Lemma 10. *There exists a distribution $\pi_3 \in \mathcal{D}'(\mathbb{R}_+^*\times\mathbb{R}^3)$, such that*

$$\nabla_\perp\cdot(\rho^\varepsilon v_\perp^\varepsilon \otimes v_\perp^\varepsilon) \longrightarrow \nabla_\perp\cdot(v_\perp\otimes v_\perp) \;+\; \nabla_\perp\pi_3 \quad \text{in} \;\; \mathcal{D}'(\mathbb{R}_+^*\times\mathbb{R}^3)\,.$$

Proof. First, we observe the following decomposition

$$\nabla_\perp\cdot(\rho^\varepsilon v_\perp^\varepsilon \otimes v_\perp^\varepsilon) = \nabla_\perp\cdot(\rho^\varepsilon v_\perp^\varepsilon \otimes \mathbb{P}_\perp v_\perp^\varepsilon) + \nabla_\perp\cdot(\mathbb{P}_\perp(\rho^\varepsilon v_\perp^\varepsilon)\otimes\mathbb{Q}_\perp v_\perp^\varepsilon) + \nabla_\perp\cdot(\mathbb{Q}_\perp(\rho^\varepsilon v_\perp^\varepsilon)\otimes\mathbb{Q}_\perp v_\perp^\varepsilon)\,. \tag{83}$$

Using the fourth and fifth statements of Lemma 8, for the first term of the right-hand side of (83), we obtain $\nabla_\perp\cdot(\rho^\varepsilon v_\perp^\varepsilon\otimes v_\perp^\varepsilon) \rightharpoonup \nabla_\perp\cdot(v_\perp\otimes v_\perp)$ in $\mathcal{D}'(\mathbb{R}_+^*\times\mathbb{R}^3)$. Using the third and seventh statements of Lemma 8, for the second term of the right-hand side of (83), we obtain $\nabla_\perp\cdot(\mathbb{P}_\perp(\rho^\varepsilon v_\perp^\varepsilon)\otimes\mathbb{Q}_\perp v_\perp^\varepsilon) \rightharpoonup 0$ in $\mathcal{D}'(\mathbb{R}_+^*\times\mathbb{R}^3)$. It remains to show that we have $\nabla_\perp\cdot(\mathbb{Q}_\perp(\rho^\varepsilon v_\perp^\varepsilon)\otimes\mathbb{Q}_\perp v_\perp^\varepsilon) \rightharpoonup \nabla_\perp\pi_3$ in $\mathcal{D}'(\mathbb{R}_+^*\times\mathbb{R}^3)$, or equivalently, that, for any $\psi_\perp \in \mathscr{C}_c^\infty(\mathbb{R}_+\times\mathbb{R}^3)$ such that $\nabla_\perp\cdot\psi_\perp = 0$, we have

$$\lim_{\varepsilon\to 0}\Gamma^\varepsilon := \lim_{\varepsilon\to 0}\int_{\mathbb{R}_+} dt \int_{\mathbb{R}^3} dx\, {}^t(\mathbb{Q}_\perp(\rho^\varepsilon v_\perp^\varepsilon))[D_\perp\psi_\perp]\mathbb{Q}_\perp v_\perp^\varepsilon = 0\,. \tag{84}$$

Since $\mathbb{Q}_\perp v_\perp^\varepsilon$ is bounded in $L^2_{\mathrm{loc}}(\mathbb{R}_+; H^1(\mathbb{R}^3))$ uniformly with respect to ε, Using (79) with $\sigma \in [\kappa, 1]$, we obtain

$$|\Gamma^\varepsilon - \Gamma_1^\varepsilon| \lesssim \omega(\varepsilon)\,, \qquad \text{with} \qquad \Gamma_1^\varepsilon := \int_{\mathbb{R}_+} dt \int_{\mathbb{R}^3} dx\, {}^t(\mathcal{S}_2(t/\varepsilon)\widetilde{\mathcal{U}}^\varepsilon)[D_\perp\psi_\perp]\mathbb{Q}_\perp v_\perp^\varepsilon\,. \tag{85}$$

Using the isometry $\mathcal{S}$ in $H^\alpha(\mathbb{R}^3)$ for any $\alpha\in\mathbb{R}$, estimate (95), bound (77), and the fact that the group $\mathcal{S}$ and the mollification operator $\mathcal{J}_{3,\delta}$ commute, we obtain, for any $\mu\in[0,1]$ and $\delta\in(0,1)$,

$$\begin{aligned}
\|\mathcal{S}_2(t/\varepsilon)\mathcal{J}_{3,\delta}^2\widetilde{\mathcal{U}}^\varepsilon - \mathcal{S}_2(t/\varepsilon)\widetilde{\mathcal{U}}^\varepsilon\|_{L^\infty_{\mathrm{loc}}(\mathbb{R}_+;H^{-\mu}(\mathbb{R}^3))} &\leq \|\mathcal{J}_{3,\delta}\mathcal{S}_2(t/\varepsilon)\widetilde{\mathcal{U}}^\varepsilon - \mathcal{S}_2(t/\varepsilon)\widetilde{\mathcal{U}}^\varepsilon\|_{L^\infty_{\mathrm{loc}}(\mathbb{R}_+;H^{-\mu}(\mathbb{R}^3))} \\
&\quad + \;\; \|\mathcal{J}_{3,\delta}\mathcal{S}_2(t/\varepsilon)\mathcal{J}_{3,\delta}\widetilde{\mathcal{U}}^\varepsilon - \mathcal{S}_2(t/\varepsilon)\mathcal{J}_{3,\delta}\widetilde{\mathcal{U}}^\varepsilon\|_{L^\infty_{\mathrm{loc}}(\mathbb{R}_+;H^{-\mu}(\mathbb{R}^3))} \\
&\lesssim \delta^\mu(1+\|\chi_\delta\|_{L^1(\mathbb{R}^3)})\|\widetilde{\mathcal{U}}^\varepsilon\|_{L^\infty_{\mathrm{loc}}(\mathbb{R}_+;L^2(\mathbb{R}^3))} \lesssim \delta^\mu\,.
\end{aligned} \tag{86}$$

Since $\mathbb{Q}_\perp v_\perp^\varepsilon$ is bounded in $L^2_{\mathrm{loc}}(\mathbb{R}_+; H^1(\mathbb{R}^3))$ uniformly with respect to ε, using (86), we obtain, for any $\mu\in(0,1]$,

$$|\Gamma_1^\varepsilon - \Gamma_2^\varepsilon| \lesssim \delta^\mu\,, \qquad \text{with} \qquad \Gamma_2^\varepsilon := \int_{\mathbb{R}_+} dt \int_{\mathbb{R}^3} dx\, {}^t(\mathcal{S}_2(t/\varepsilon)\mathcal{J}_{3,\delta}^2\widetilde{\mathcal{U}}^\varepsilon)[D_\perp\psi_\perp]\mathbb{Q}_\perp v_\perp^\varepsilon\,. \tag{87}$$

Since for any $\delta > 0$, and any $\nu \geq 0$, $\mathcal{S}_2(t/\varepsilon)\mathcal{J}_{3,\delta}^2 \widetilde{\mathcal{U}}^\varepsilon \in L^\infty_{\rm loc}(\mathbb{R}_+; H^\nu(\mathbb{R}^3))$, there exists a constant C_δ such that $\|\mathcal{S}_2(t/\varepsilon)\mathcal{J}_{3,\delta}^2 \widetilde{\mathcal{U}}^\varepsilon\|_{L^\infty_{\rm loc}(\mathbb{R}_+;H^\sigma(\mathbb{R}^3))} \leq C_\delta$, where C_δ explodes as $\delta \to 0$. Then, using (80), we obtain

$$|\Gamma_2^\varepsilon - \Gamma_3^\varepsilon| \lesssim C_\delta(\omega(\varepsilon) + \varepsilon), \quad \text{with} \quad \Gamma_3^\varepsilon := \int_{\mathbb{R}_+} dt \int_{\mathbb{R}^3} dx\, {}^t(\mathcal{S}_2(t/\varepsilon)\mathcal{J}_{3,\delta}^2 \widetilde{\mathcal{U}}^\varepsilon)[D_\perp\psi_\perp]\mathcal{S}_2(t/\varepsilon)\widetilde{\mathcal{U}}^\varepsilon . \tag{88}$$

We now claim that

$$|\Gamma_3^\varepsilon - \Gamma_4^\varepsilon| \lesssim \delta, \quad \text{with} \quad \Gamma_4^\varepsilon := \int_{\mathbb{R}_+} dt \int_{\mathbb{R}^3} dx\, {}^t(\mathcal{S}_2(t/\varepsilon)\mathcal{J}_{3,\delta} \widetilde{\mathcal{U}}^\varepsilon)[D_\perp\psi_\perp]\mathcal{S}_2(t/\varepsilon)\mathcal{J}_{3,\delta}\widetilde{\mathcal{U}}^\varepsilon . \tag{89}$$

Indeed, using in order $[\mathcal{S}\,,\,\mathcal{J}_{3,\delta}] = 0$ (where $[\,\cdot\,,\,\cdot\,]$ is the commutator), $L^1 * L^2 \subset L^2$, Cauchy–Scwharz inequality, $\|x\chi(x)\|_{L^1(\mathbb{R}^3)} \leq C < \infty$, estimate (95) (with $\sigma = s = 0$), the isometry $\mathcal{S}$ in $H^\alpha(\mathbb{R}^3)$ for any $\alpha \in \mathbb{R}$, and bound (77), we obtain

$$\begin{aligned}
\Gamma_3^\varepsilon - \Gamma_4^\varepsilon &= \int_{\mathbb{R}_+} dt \int_{\mathbb{R}^3} dx\, {}^t(\mathcal{S}_2(t/\varepsilon)\mathcal{J}_{3,\delta} \widetilde{\mathcal{U}}^\varepsilon)[\mathcal{J}_{3,\delta}\,,\, D_\perp\psi_\perp]\mathcal{S}_2(t/\varepsilon)\widetilde{\mathcal{U}}^\varepsilon \\
&= \int_{\mathbb{R}_+} dt \int_{\mathbb{R}^3} dx\, {}^t(\mathcal{S}_2(t/\varepsilon)\mathcal{J}_{3,\delta} \widetilde{\mathcal{U}}^\varepsilon)(x) \int_{\mathbb{R}^3} dy\, \big(D_\perp\psi_\perp(y) - D_\perp\psi_\perp(x)\big)\chi_\delta(x-y)\mathcal{S}_2(t/\varepsilon)\widetilde{\mathcal{U}}^\varepsilon(y) \\
&\leq \delta \|D_\perp^2\psi_\perp\|_{L^\infty(\mathbb{R}_+\times\mathbb{R}^3)} \|\mathcal{S}_2(t/\varepsilon)\mathcal{J}_{3,\delta}\widetilde{\mathcal{U}}^\varepsilon\|_{L^\infty_{\rm loc}(\mathbb{R}_+;L^2(\mathbb{R}^3))} \|\,|\tfrac{x}{\delta}\chi_\delta| * (\mathcal{S}_2(t/\varepsilon)\widetilde{\mathcal{U}}^\varepsilon)\|_{L^\infty_{\rm loc}(\mathbb{R}_+;L^2(\mathbb{R}^3))} \\
&\lesssim \delta \|\mathcal{S}_2(t/\varepsilon)\widetilde{\mathcal{U}}^\varepsilon\|^2_{L^\infty_{\rm loc}(\mathbb{R}_+;L^2(\mathbb{R}^3))} \lesssim \delta \|\widetilde{\mathcal{U}}^\varepsilon\|^2_{L^\infty_{\rm loc}(\mathbb{R}_+;L^2(\mathbb{R}^3))} \lesssim \delta .
\end{aligned}$$

We now claim that there exists a constant $\widetilde{C}_\delta$, which explodes as $\delta \to 0$, such that for all $\nu \geq 0$,

$$\Gamma^\varepsilon := \|\mathcal{J}_{3,\delta}\widetilde{\mathcal{U}}^\varepsilon(t_1) - \mathcal{J}_{3,\delta}\widetilde{\mathcal{U}}^\varepsilon(t_2)\|_{H^\nu(\mathbb{R}^3)} \leq \widetilde{C}_\delta\big(|t_1 - t_2| + \omega(\varepsilon)\big) \tag{90}$$

Indeed, using (76) and (78), we obtain

$$\begin{aligned}
\Gamma^\varepsilon &\leq \|\chi_\delta\|_{H^{\nu+r}} \|\mathcal{U}^\varepsilon(t_1) - \mathcal{U}^\varepsilon(t_2)\|_{H^{-r}(\mathbb{R}^3)} + 2\|\chi_\delta\|_{H^{\nu+\sigma}} \|\widetilde{\mathcal{U}}^\varepsilon - \mathcal{U}^\varepsilon\|_{L^\infty_{\rm loc}(\mathbb{R}_+;H^{-\sigma}(\mathbb{R}^3))} \\
&\leq \widetilde{C}_\delta \Big(\int_{t_1}^{t_2} d\tau\, \|\partial_t \mathcal{U}^\varepsilon(\tau)\|_{H^{-r}(\mathbb{R}^3)} + \omega(\varepsilon) \Big) \leq \widetilde{C}_\delta\big(|t_1 - t_2| + \omega(\varepsilon)\big) .
\end{aligned}$$

Time continuity estimate (90) allows us to replace the term $\mathcal{S}(t/\varepsilon)\mathcal{J}_{3,\delta}\widetilde{\mathcal{U}}^\varepsilon$ by its time regularization $\mathcal{J}_{1,\eta}\mathcal{S}_2(t/\varepsilon)\mathcal{J}_{3,\delta}\widetilde{\mathcal{U}}^\varepsilon$ (with $\eta > 0$) since the error is controlled as

$$\|\mathcal{S}(t/\varepsilon)\mathcal{J}_{1,\eta}\mathcal{J}_{3,\delta}\widetilde{\mathcal{U}}^\varepsilon - \mathcal{S}(t/\varepsilon)\mathcal{J}_{3,\delta}\widetilde{\mathcal{U}}^\varepsilon\|_{L^\infty_{\rm loc}(\mathbb{R}_+;H^\nu(\mathbb{R}^3))} \leq \widetilde{C}_\delta\big(\eta + \omega(\varepsilon)\big) , \tag{91}$$

for all $\nu \geq 0$. Using (91), we then obtain

$$|\Gamma_4^\varepsilon - \Gamma_5^\varepsilon| \lesssim \widetilde{C}_\delta\big(\eta + \omega(\varepsilon)\big) , \quad \text{with} \quad \Gamma_5^\varepsilon := \int_{\mathbb{R}_+} dt \int_{\mathbb{R}^3} dx\, {}^t(\mathcal{S}_2(t/\varepsilon)\mathcal{J}_{1,\eta}\mathcal{J}_{3,\delta}\widetilde{\mathcal{U}}^\varepsilon)[D_\perp\psi_\perp]\mathcal{S}_2(t/\varepsilon)\mathcal{J}_{1,\eta}\mathcal{J}_{3,\delta}\widetilde{\mathcal{U}}^\varepsilon . \tag{92}$$

Therefore, by gathering estimates (87), (88), (89) and (92), and by first taking the limit $\varepsilon \to 0$, then the limit $\eta \to 0$, and finally the limit $\delta \to 0$, we observe that the proof of Lemma 10 is complete, if we prove $\Gamma_5^\varepsilon \to 0$, as $\varepsilon \to 0$. This is the matter of the rest of the proof.

In other words, we just have to show $\lim_{\varepsilon\to 0}\Gamma_5^\varepsilon = 0$, when $\mathcal{J}_{1,\eta}\mathcal{J}_{3,\delta}\widetilde{\mathcal{U}}^\varepsilon$ is replaced by a smooth version of $\widetilde{\mathcal{U}}^\varepsilon$, that we denote by $\mathcal{V}^\varepsilon$ (to simplify the notation) with $\mathcal{V}^\varepsilon \in \mathscr{C}^m(\mathbb{R}_+; H^\nu(\mathbb{R}^3))$ for any $m \geq 0$ and any $\nu \geq 0$ (not uniformly with respect to η and δ). As in the proof of Lemma 5, following the spirit of the proof of the convergence result in the part III of [34], we introduce $\mathfrak{U}^\varepsilon \equiv {}^t(\phi^\varepsilon, {}^t\Phi^\varepsilon) := {}^t(\mathcal{S}_1(t/\varepsilon)\,\mathcal{V}^\varepsilon, {}^t\mathcal{S}_2(t/\varepsilon)\,\mathcal{V}^\varepsilon) = \mathcal{S}(t/\varepsilon)\,\mathcal{V}^\varepsilon$, with $\mathcal{V}^\varepsilon = {}^t(\psi^\varepsilon, {}^t\Psi^\varepsilon) \in \mathscr{C}^m(\mathbb{R}_+; H^\nu(\mathbb{R}^3))$, and we compute explicity $\nabla_\perp \cdot (\Phi^\varepsilon \otimes \Phi^\varepsilon)$ via Fourier transform. Since $\mathbb{P}_\perp \Psi^\varepsilon = 0$, then $\Psi^\varepsilon = \nabla_\perp \psi^\varepsilon$, with $\psi^\varepsilon = \Delta_\perp^{-1}\nabla_\perp \cdot \Psi^\varepsilon$. This and commutation between $\mathcal{S}$ and $\mathbb{P}_\perp$ imply $\mathbb{P}_\perp \Phi^\varepsilon = 0$, so that $\Phi^\varepsilon = \nabla_\perp \varphi^\varepsilon$, with $\varphi^\varepsilon = \Delta_\perp^{-1}\nabla_\perp \cdot \Phi^\varepsilon$. We introduce the Fourier decompositions

$$\psi^\varepsilon = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} d\xi\, e^{i\xi\cdot x}\,\hat{\psi}^\varepsilon(t,\xi)\,, \quad \Psi^\varepsilon = \frac{i}{(2\pi)^3}\int_{\mathbb{R}^3} d\xi\, e^{i\xi\cdot x}\,\hat{\psi}^\varepsilon(t,\xi)\xi_\perp\,,$$

and we denote by $\hat{\mathfrak{U}}^\varepsilon = {}^t(\hat{\phi}^\varepsilon, {}^t\hat{\Phi}^\varepsilon = i\,{}^t\xi_\perp\hat{\varphi}^\varepsilon)$ the Fourier transform of $\mathfrak{U}^\varepsilon \equiv {}^t(\phi^\varepsilon, {}^t\Phi^\varepsilon = \nabla_\perp\varphi^\varepsilon)$. Inserting the Fourier decomposition of $\mathfrak{U}^\varepsilon$ in the linear equation $\partial_t\,\mathfrak{U}^\varepsilon = \mathcal{L}\,\mathfrak{U}^\varepsilon/\varepsilon$, we are led to solve linear second-order ODEs in time for the Fourier coefficients $\hat{\phi}^\varepsilon(t)$ and $\hat{\varphi}^\varepsilon(t)$, with the inital conditions $\hat{\mathfrak{U}}^\varepsilon(0) = \hat{\mathcal{V}}^\varepsilon(t)$ and $\partial_t\,\hat{\mathfrak{U}}^\varepsilon(0) = \hat{\mathcal{L}}\,\hat{\mathcal{V}}^\varepsilon(t)/\varepsilon$, where $\hat{\mathcal{L}} = i\,{}^t(-c\,\xi_\perp\cdot\,, {}^t\xi_\perp)$. Solving these linear ODEs, we obtain for Φ^ε,

$$\Phi^\varepsilon = \nabla_\perp\varphi^\varepsilon = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} d\xi\, e^{i\xi\cdot x}\,\frac{i\xi_\perp}{|\xi_\perp|}\left\{\hat{\mathfrak{m}}^\varepsilon(t,\xi)\cos\left(\sqrt{c}|\xi_\perp|\tfrac{t}{\varepsilon}\right) - \tfrac{1}{\sqrt{c}}\hat{\psi}(t,\xi)\sin\left(\sqrt{c}|\xi_\perp|\tfrac{t}{\varepsilon}\right)\right\},$$

where we have introduced $\hat{\mathfrak{m}}^\varepsilon = \hat{\psi}^\varepsilon|\xi_\perp|$ (with $\mathfrak{m}^\varepsilon \in \mathscr{C}^m(\mathbb{R}_+; H^\nu(\mathbb{R}^3))$, $\forall\,(m,\nu) \geq 0$) to symmetrize the expressions. We then obtain

$$\Phi^\varepsilon \otimes \Phi^\varepsilon = -\frac{1}{2(2\pi)^6}\int_{\mathbb{R}^3} d\xi \int_{\mathbb{R}^3} d\zeta\, e^{i(\xi+\zeta)\cdot x}\,\theta^\varepsilon(t,\xi)\theta^\varepsilon(t,\zeta)(\xi_\perp\otimes\zeta_\perp + \zeta_\perp\otimes\xi_\perp)\,, \tag{93}$$

with $\theta^\varepsilon(t,\xi) = \big(\hat{\mathfrak{m}}^\varepsilon(t,\xi)/|\xi_\perp|\big)\cos(\sqrt{c}|\xi_\perp|t/\varepsilon) - \big(\hat{\psi}(t,\xi)/(\sqrt{c}|\xi_\perp|)\big)\sin(\sqrt{c}|\xi_\perp|t/\varepsilon)$. Then, we obtain

$$\begin{aligned}\nabla_\perp\cdot(\Phi^\varepsilon\otimes\Phi^\varepsilon) = &-\frac{i}{4(2\pi)^6}\int_{\mathbb{R}^3} d\xi\int_{\mathbb{R}^3} d\zeta\, e^{i(\xi+\zeta)\cdot x}\,(\xi_\perp+\zeta_\perp)|\xi_\perp+\zeta_\perp|^2\theta^\varepsilon(t,\xi)\theta^\varepsilon(t,\zeta)\\ &+\frac{i}{4(2\pi)^6}\int_{\mathbb{R}^3} d\xi\int_{\mathbb{R}^3} d\zeta\, e^{i(\xi+\zeta)\cdot x}\,(\xi_\perp-\zeta_\perp)(|\xi_\perp|^2-|\zeta_\perp|^2)\theta^\varepsilon(t,\xi)\theta^\varepsilon(t,\zeta)\,.\end{aligned} \tag{94}$$

The first term of the right-hand side of (94), is a gradient and thus its contribution in Γ_5^ε is null since $\nabla_\perp\cdot\psi_\perp = 0$. In fact, following estimates below, we can show that this term converges in $\mathcal{D}'(\mathbb{R}_+^*\times\mathbb{R}^3)$ to the pressure term $\nabla_\perp\pi_3$, as $\varepsilon\to 0$. Then, it remains to show that the second term of the right-hand side of (94) vanishes in $\mathcal{D}'(\mathbb{R}_+^*\times\mathbb{R}^3)$, as $\varepsilon\to 0$. In fact, because of the presence of the factor $1/|\xi_\perp|$ in the definition of θ^ε and since, contrary to the periodic case, we now have $\hat{\psi}(t,0)\neq 0$, we have to consider a truncated version of $\Phi^\varepsilon\otimes\Phi^\varepsilon$ around low frequences ($\xi_\perp\simeq 0$). For this, for any $\delta\in(0,1)$, we consider ψ_δ^ε, ψ_δ^ε, $\mathfrak{m}_\delta^\varepsilon$, and Φ_δ^ε defined by inverse Fourier transforms of $\hat{\psi}_\delta^\varepsilon := \hat{\psi}^\varepsilon\mathbb{1}_{\{|\xi_\perp|\geq\delta\}}$, $\hat{\psi}_\delta^\varepsilon = \hat{\psi}^\varepsilon\mathbb{1}_{\{|\xi_\perp|\geq\delta\}}$, $\hat{\mathfrak{m}}_\delta^\varepsilon := |\xi_\perp|\hat{\psi}_\delta^\varepsilon$ and by $\Phi_\delta^\varepsilon := \mathcal{S}_2(t/\varepsilon)\,{}^t(\psi_\delta^\varepsilon, {}^t\nabla_\perp\psi_\delta^\varepsilon)$. Using Cauchy–Schwarz inequality, we then obtain the error term

$$\begin{aligned}&\|\Phi^\varepsilon\otimes\Phi^\varepsilon - \Phi_\delta^\varepsilon\otimes\Phi_\delta^\varepsilon\|_{L^\infty_{\mathrm{loc}}(\mathbb{R}_+;L^\infty(\mathbb{R}^3))}\\ &\lesssim \int_{\mathbb{R}^3} d\xi\int_{\mathbb{R}^3} d\zeta\,(|\hat{\mathfrak{m}}^\varepsilon(t,\xi)| + |\hat{\psi}^\varepsilon(t,\xi)|)\mathbb{1}_{\{|\xi_\perp|<\delta\}}\,(|\hat{\mathfrak{m}}^\varepsilon(t,\zeta)| + |\hat{\psi}^\varepsilon(t,\zeta)|)\frac{(1+|\zeta|^2)^{\nu/2}}{(1+|\zeta|^2)^{\nu/2}}\end{aligned}$$

$$\lesssim \int_{\mathbb{R}} d\xi_\parallel \Big(\int_{|\xi_\perp|\le\delta} d\xi_\perp \Big)^{1/2} \Big(\int_{\mathbb{R}^2} d\xi_\perp \, (|\hat{\mathfrak{m}}^\varepsilon(t,\xi)|^2 + |\hat{\psi}^\varepsilon(t,\xi)|^2) \Big)^{1/2}$$

$$\Big(\int_{\mathbb{R}^3} d\zeta \, (1+|\zeta|^2)^\nu \, (|\hat{\mathfrak{m}}^\varepsilon(t,\zeta)|^2 + |\hat{\psi}^\varepsilon(t,\zeta)|^2) \Big)^{1/2} \Big(\int_{\mathbb{R}^3} d\zeta \, (1+|\zeta|^2)^{-\nu} \Big)^{1/2}$$

$$\lesssim \delta \int_{\mathbb{R}} d\xi_\parallel (1+|\xi_\parallel|^2)^{-\mu/2} \Big(\int_{\mathbb{R}^2} d\xi_\perp \, (1+|\xi_\parallel|^2)^\mu \, (|\hat{\mathfrak{m}}^\varepsilon(t,\xi)|^2 + |\hat{\psi}^\varepsilon(t,\xi)|^2) \Big)^{1/2}$$

$$\big(\|\mathfrak{m}^\varepsilon\|_{L^\infty_{\mathrm{loc}}(\mathbb{R}_+;H^\nu(\mathbb{R}^3))} + \|\psi^\varepsilon\|_{L^\infty_{\mathrm{loc}}(\mathbb{R}_+;H^\nu(\mathbb{R}^3))} \big)$$

$$\lesssim \delta \big(\|\mathfrak{m}^\varepsilon\|_{L^\infty_{\mathrm{loc}}(\mathbb{R}_+;H^\mu(\mathbb{R}^3))} + \|\psi^\varepsilon\|_{L^\infty_{\mathrm{loc}}(\mathbb{R}_+;H^\mu(\mathbb{R}^3))} \big) \big(\|\mathfrak{m}^\varepsilon\|_{L^\infty_{\mathrm{loc}}(\mathbb{R}_+;H^\nu(\mathbb{R}^3))} + \|\psi^\varepsilon\|_{L^\infty_{\mathrm{loc}}(\mathbb{R}_+;H^\nu(\mathbb{R}^3))} \big) \lesssim \delta \,,$$

for any $\mu > 1/2$ and $\nu > 3/2$. Therefore, we just have to show the claim for Φ^ε_δ with any fixed $\delta \in (0,1)$. This is equivalent to assuming that $\hat{\psi}^\varepsilon$ and $\hat{\mathfrak{m}}^\varepsilon$ vanish in the tube $\mathscr{T}_\delta(\xi) := \{\xi \in \mathbb{R}^3 \mid |\xi_\perp| < \delta\}$, for any fixed $\delta \in (0,1)$, and independently of $(t, \xi_\parallel, \varepsilon)$. With this assumption, we just need to estimate the second term of the right-hand side of (94) that we decompose as follows,

$$\int_{\mathbb{R}^3} d\xi \int_{\mathbb{R}^3} d\zeta \, e^{\mathrm{i}(\xi+\zeta)\cdot x} \, (\xi_\perp - \zeta_\perp)(|\xi_\perp|^2 - |\zeta_\perp|^2) \theta^\varepsilon(t,\xi)\theta^\varepsilon(t,\zeta)$$

$$= \frac{1}{2} \int_{\mathbb{R}^3} d\xi \int_{\mathbb{R}^3} d\zeta \, e^{\mathrm{i}(\xi+\zeta)\cdot x} \, (\xi_\perp - \zeta_\perp)(|\xi_\perp|^2 - |\zeta_\perp|^2) \Big\{$$

$$\Big[\cos\big(\sqrt{c}(|\xi_\perp| + |\zeta_\perp|)\tfrac{t}{\varepsilon}\big) + \cos\big(\sqrt{c}(|\xi_\perp| - |\zeta_\perp|)\tfrac{t}{\varepsilon}\big) \Big] \frac{\hat{\mathfrak{m}}^\varepsilon(t,\xi)\hat{\mathfrak{m}}^\varepsilon(t,\zeta)}{|\xi_\perp||\zeta_\perp|}$$

$$- \Big[\sin\big(\sqrt{c}(|\xi_\perp| + |\zeta_\perp|)\tfrac{t}{\varepsilon}\big) + \sin\big(\sqrt{c}(|\xi_\perp| - |\zeta_\perp|)\tfrac{t}{\varepsilon}\big) \Big] \frac{\hat{\psi}^\varepsilon(t,\xi)\hat{\mathfrak{m}}^\varepsilon(t,\zeta)}{\sqrt{c}|\xi_\perp||\zeta_\perp|}$$

$$- \Big[\sin\big(\sqrt{c}(|\xi_\perp| + |\zeta_\perp|)\tfrac{t}{\varepsilon}\big) - \sin\big(\sqrt{c}(|\xi_\perp| - |\zeta_\perp|)\tfrac{t}{\varepsilon}\big) \Big] \frac{\hat{\mathfrak{m}}^\varepsilon(t,\xi)\hat{\psi}^\varepsilon(t,\zeta)}{\sqrt{c}|\xi_\perp||\zeta_\perp|}$$

$$- \Big[\cos\big(\sqrt{c}(|\xi_\perp| + |\zeta_\perp|)\tfrac{t}{\varepsilon}\big) - \cos\big(\sqrt{c}(|\xi_\perp| - |\zeta_\perp|)\tfrac{t}{\varepsilon}\big) \Big] \frac{\hat{\psi}^\varepsilon(t,\xi)\hat{\psi}^\varepsilon(t,\zeta)}{c|\xi_\perp||\zeta_\perp|} \Big\}.$$

The eight terms in the right-hand side of the previous equation can be estimated in a similar way, hence we only treat one of them, for instance,

$$\int_{\mathbb{R}^3} d\xi \int_{\mathbb{R}^3} d\zeta \, e^{\mathrm{i}(\xi+\zeta)\cdot x} \, (\xi_\perp - \zeta_\perp)(|\xi_\perp|^2 - |\zeta_\perp|^2) \sin\big(\sqrt{c}(|\xi_\perp| - |\zeta_\perp|)\tfrac{t}{\varepsilon}\big) \frac{\hat{\psi}^\varepsilon(t,\xi)\hat{\mathfrak{m}}^\varepsilon(t,\zeta)}{\sqrt{c}|\xi_\perp||\zeta_\perp|} = \partial_t \mathcal{T}^\varepsilon_1 + \mathcal{T}^\varepsilon_2 \,,$$

where,

$$\mathcal{T}^\varepsilon_1 = -\varepsilon \int_{\mathbb{R}^3} d\xi \int_{\mathbb{R}^3} d\zeta \, e^{\mathrm{i}(\xi+\zeta)\cdot x} \, (\xi_\perp - \zeta_\perp)(|\xi_\perp| + |\zeta_\perp|) \cos\big(\sqrt{c}(|\xi_\perp| - |\zeta_\perp|)\tfrac{t}{\varepsilon}\big) \frac{\hat{\psi}^\varepsilon(t,\xi)\hat{\mathfrak{m}}^\varepsilon(t,\zeta)}{c|\xi_\perp||\zeta_\perp|} \,,$$

and

$$\mathcal{T}^\varepsilon_2 = \varepsilon \int_{\mathbb{R}^3} d\xi \int_{\mathbb{R}^3} d\zeta \, e^{\mathrm{i}(\xi+\zeta)\cdot x} \, (\xi_\perp - \zeta_\perp)(|\xi_\perp| + |\zeta_\perp|) \cos\big(\sqrt{c}(|\xi_\perp| - |\zeta_\perp|)\tfrac{t}{\varepsilon}\big) \frac{\partial}{\partial t} \Big(\frac{\hat{\psi}^\varepsilon(t,\xi)\hat{\mathfrak{m}}^\varepsilon(t,\zeta)}{c|\xi_\perp||\zeta_\perp|} \Big) \,,$$

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The proof of Lemma 10 will be complete if we prove that $\|\mathcal{T}_1^\varepsilon\|_{L^\infty_{\mathrm{loc}}(\mathbb{R}_+;L^\infty(\mathbb{R}^3))}$ and $\|\mathcal{T}_2^\varepsilon\|_{L^\infty_{\mathrm{loc}}(\mathbb{R}_+;L^\infty(\mathbb{R}^3))}$ vanish as $\varepsilon \to 0$. Indeed, for any $\nu > 3/2$, using Cauchy–Schwarz inequality, we obtain, for any $T \in (0,+\infty)$ and any fixed $\delta \in (0,1)$,

$$\begin{aligned}
\|\mathcal{T}_1^\varepsilon\|_{L^\infty_{\mathrm{loc}}(\mathbb{R}_+;L^\infty(\mathbb{R}^3))} &\lesssim \varepsilon \sup_{t\in[0,T]} \int_{\mathscr{T}_\delta^c(\xi)} d\xi \int_{\mathscr{T}_\delta^c(\zeta)} d\zeta \, (|\xi_\perp|^2 + |\zeta_\perp|^2) \frac{|\hat{\psi}^\varepsilon(t,\xi)|}{|\xi_\perp|} \frac{|\hat{\mathfrak{m}}^\varepsilon(t,\zeta)|}{|\zeta_\perp|} \\
&\lesssim \varepsilon\delta^{-1} \sup_{t\in[0,T]} \Big(\int_{\mathbb{R}^3} d\xi \, \frac{1}{(1+|\xi|^2)^{\nu/2}} (1+|\xi|^2)^{(\nu+1)/2} |\hat{\psi}^\varepsilon(t,\xi)| \Big) \\
&\qquad \Big(\int_{\mathbb{R}^3} d\zeta \, \frac{1}{(1+|\zeta|^2)^{\nu/2}} (1+|\zeta|^2)^{(\nu+1)/2} |\hat{\mathfrak{m}}^\varepsilon(t,\zeta)| \Big) \\
&\lesssim \varepsilon\delta^{-1} \|\psi^\varepsilon\|_{L^\infty_{\mathrm{loc}}(\mathbb{R}_+;H^{\nu+1}(\mathbb{R}^3))} \|\mathfrak{m}^\varepsilon\|_{L^\infty_{\mathrm{loc}}(\mathbb{R}_+;H^{\nu+1}(\mathbb{R}^3))} \lesssim \varepsilon \,,
\end{aligned}$$

where $\mathscr{T}_\delta^c(\xi)$ denotes the complementary set of $\mathscr{T}_\delta(\xi)$. In the same way that we have controlled the term $\mathcal{T}_1^\varepsilon$, we obtain the following estimate for the term $\mathcal{T}_2^\varepsilon$,

$$\begin{aligned}
\|\mathcal{T}_2^\varepsilon\|_{L^\infty_{\mathrm{loc}}(\mathbb{R}_+;L^\infty(\mathbb{R}^3))} &\lesssim \varepsilon\delta^{-1} \big(\|\partial_t\psi^\varepsilon\|_{L^\infty_{\mathrm{loc}}(\mathbb{R}_+;H^{\nu+1}(\mathbb{R}^3))} \|\mathfrak{m}^\varepsilon\|_{L^\infty_{\mathrm{loc}}(\mathbb{R}_+;H^{\nu+1}(\mathbb{R}^3))} \\
&\qquad + \|\psi^\varepsilon\|_{L^\infty_{\mathrm{loc}}(\mathbb{R}_+;H^{\nu+1}(\mathbb{R}^3))} \|\partial_t\mathfrak{m}^\varepsilon\|_{L^\infty_{\mathrm{loc}}(\mathbb{R}_+;H^{\nu+1}(\mathbb{R}^3))} \lesssim \varepsilon \,. \qquad \square
\end{aligned}$$

Appendix A. Toolbox

This section collects lemmas frequently used throughout the text. We start with inequalities for the first-order Taylor expansion of the power function $x \mapsto x^\gamma$, with $\gamma > 1$. These inequalities, mentioned in [34], can be seen as "generalized" strong convexity properties of the power function on the non-negative half-line.

Lemma 11 *(Convexity properties of the power function). Let $\bar{x} > 0$ and $\gamma > 1$ be fixed positive real numbers. For all $R \in]\bar{x}, +\infty[$, there exist positive constants ν_i, $i = 1,2,3$, depending on γ, R and $\bar{x}$, such that*

$$\begin{aligned}
x^\gamma - \gamma x \bar{x}^{\gamma-1} + (\gamma-1)\bar{x}^\gamma \\
= x^\gamma - \bar{x}^\gamma - \gamma\bar{x}^{\gamma-1}(x-\bar{x}) \geq \begin{cases} \nu_1 \, |x-\bar{x}|^2, & 0 \leq x, \quad \gamma \geq 2\,, \\ \nu_2 \, |x-\bar{x}|^2, & 0 \leq x \leq R, \quad 1 < \gamma < 2\,, \\ \nu_3 \, |x-\bar{x}|^\gamma, & R < x, \quad 1 < \gamma < 2\,. \end{cases}
\end{aligned}$$

Proof. Knowing that $\gamma > 1$, the power function $\mathbb{R}^+ \ni x \mapsto x^\gamma \in \mathbb{R}^+$ is convex on $\mathbb{R}^+$, and strongly convex on any compact set of $\mathbb{R}_+^*$. The details are left to the reader. $\square$

Recall that the Orlicz space $L_2^\gamma(\mathbb{R}^3)$ is defined by (34), or see Appendix A in [33].

Lemma 12 *(A criterion for belonging to the Orlicz spaces $L_2^\gamma(\Omega)$). Let $\gamma > 1$, $\bar{f} > 0$ and $\delta > 0$ be fixed positive real numbers. Let $f \in L^\gamma_{\mathrm{loc}}(\mathbb{R}^3;\mathbb{R}_+)$ be given. Define*

$$\begin{aligned}
&\Pi_{\bar{f},\gamma} : L^\gamma_{\mathrm{loc}}(\mathbb{R}^3;\mathbb{R}_+) \to L^1_{\mathrm{loc}}(\mathbb{R}^3;\mathbb{R}_+)\,, \qquad \Pi_{\bar{f},\gamma}(f) := f^\gamma - \gamma\bar{f}^{\gamma-1} f + (\gamma-1)\bar{f}^\gamma\,, \\
&\mathfrak{Z}_{2,\delta}^{\gamma,\bar{f}} : L^\gamma_{\mathrm{loc}}(\mathbb{R}^3;\mathbb{R}_+) \to L^1_{\mathrm{loc}}(\mathbb{R}^3;\mathbb{R}_+)\,, \qquad \mathfrak{Z}_{2,\delta}^{\gamma,\bar{f}}(f) := |f-\bar{f}|^2 \mathbb{1}_{\{|f-\bar{f}|\leq\delta\}} + |f-\bar{f}|^\gamma \mathbb{1}_{\{|f-\bar{f}|>\delta\}}\,.
\end{aligned}$$

There exist two constants κ_1 and κ_2, depending on $(\bar{f},\gamma,\delta)$, such that

$$\kappa_1 \, \mathfrak{Z}_{2,\delta}^{\gamma,\bar{f}}(f) \leq \Pi_{\bar{f},\gamma}(f) \leq \kappa_2 \, \mathfrak{Z}_{2,\delta}^{\gamma,\bar{f}}(f)\,.$$

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It follows that $\Pi_{\bar{f},\gamma}(f) \in L^1(\mathbb{R}^3)$ *if and only if* $(f-\bar{f}) \in L_2^\gamma(\mathbb{R}^3)$.

Proof. The proof is long but straightforward. It mainly relies on Lemma 11, a Taylor formula with integral remainder, and on the convexity (resp. strong convexity) of $x \mapsto x^\gamma$ on $\mathbb{R}^+$ (resp. on any compact set of $\mathbb{R}^+$). The details are left to the reader. Note that this result is similar to the more general Lemma 5.3 in [33] to which we can also refer the reader for a proof. □

Lemma 13 *(A space-time compactness lemma of Simon [47]). Let* $\mathfrak{B}_0 \Subset \mathfrak{B} \subset \mathfrak{B}_1$ *be Banach spaces (the embedding* $\mathfrak{B}_0 \Subset \mathfrak{B}$ *is compact and the embedding* $\mathfrak{B} \subset \mathfrak{B}_1$ *is continuous). Let* I *be a compact interval. Fix* q *with* $1 < q \leq \infty$. *Let* $f_\varepsilon : I \to \mathfrak{B}$ *be a family of functions indexed by* ε *in a directed set*[2] J. *Thus, for all* $t \in I$, *we have* $f_\varepsilon(t) \in \mathfrak{B}$. *We assume that* $\{f_\varepsilon\}_{\varepsilon\in J}$ *is bounded uniformly with respect to* ε *in* $L^q(I;\mathfrak{B}) \cap L^1(I;\mathfrak{B}_0)$ *and that* $\{\partial_t f_\varepsilon\}_{\varepsilon\in J}$ *is bounded uniformly with respect to* ε *in* $L^1(I;\mathfrak{B}_1)$. *Then* $\{f_\varepsilon\}_{\varepsilon\in J}$ *is relatively compact in* $L^p(I;\mathfrak{B})$ *for all* p *with* $1 \leq p < q$.

We continue with the following space-time compactness lemma established and used in [33] for the proof of existence of global weak solutions to the compressible Navier–Stokes equations.

Lemma 14 *(A space-time compactness lemma of P.-L. Lions [33]). Let* Ω *be* $\mathbb{T}^N$ *or* $\mathbb{R}^N$ *or an open set of* $\mathbb{R}^N$. *Let* J *be a directed set. Let* $\{g_\varepsilon\}_{\varepsilon\in J}$, *and* $\{h_\varepsilon\}_{\varepsilon\in J}$ *be sequences converging weakly to* g *and* h, *respectively in* $L^{p_1}_{\mathrm{loc}}(\mathbb{R}_+;L^{p_2}(\Omega))$ *and* $L^{q_1}_{\mathrm{loc}}(\mathbb{R}_+;L^{q_2}(\Omega))$, *where* $1 \leq p_1,\, p_2 \leq \infty$ *and*

$$\frac{1}{p_1}+\frac{1}{q_1}=1\,,\quad \frac{1}{p_2}+\frac{1}{q_2}=1\,.$$

Above, weak convergences are weak–∗ convergences whenever some of the exponents are infinite. Assume, in addition, that

$$(i)\ \partial_t g_\varepsilon \text{ is bounded in } L^1_{\mathrm{loc}}(\mathbb{R}_+;W^{-\alpha,1}(\Omega)) \text{ for some } \alpha \geq 0\,,\ \text{uniformly in } \varepsilon\,.$$

$$(ii)\ \|h_\varepsilon(t,\cdot)-h_\varepsilon(t,\cdot+\xi)\|_{L^{q_1}_{\mathrm{loc}}(\mathbb{R}_+;L^{q_2}(\Omega))} \longrightarrow 0 \text{ as } |\xi| \to 0\,,\ \text{uniformly in } \varepsilon\,.$$

Then, the sequence $\{g_\varepsilon h_\varepsilon\}_{\varepsilon\in J}$ *converges to* gh *in the sense of distributions in* $\mathbb{R}_+^* \times \Omega$.

Proof. This is Lemma 5.1 of [33]. The assumption (i) is reminiscent to the Aubin–Lions theorem. The assumption (ii) is reminiscent to the Kolmogorov–Riesz–Fréchet criterion (e.g., see Theorem 4.26 in [7]) for compactness ("L^p-versions" of the Ascoli–Arzelà theorem). □

We end with results about mollifiers.

Lemma 15 *(Mollification operators). Let* $\chi : \mathbb{R}^d \mapsto \mathbb{R}_+$ *be a non-negative function belonging to* $\mathscr{C}_c^\infty(\mathbb{R}^d;\mathbb{R}_+)$, *and with total mass one. For any* $\eta \in (0,1)$, *define* $\chi_\eta(\cdot) = \eta^{-d}\chi(\cdot/\eta)$. *The familly of non-negative functions of mass one* $\{\chi_\eta\}_{\eta>0}$ *are called a family of mollifiers, while the operator* $\mathcal{J}_{d,\eta} : \mathcal{D}'(\mathbb{R}^d) \mapsto \mathscr{C}^\infty(\mathbb{R}^d)$, *defined as* $\mathcal{J}_{d,\eta} f = \chi_\eta * f$, *for any distribution* f, *is called a mollification operator. The mollification operator* $\mathcal{J}_{d,\eta}$ *has the following approximation property. For all* $f \in H^s(\mathbb{R}^d)$, *with* $s \in \mathbb{R}$ *and any* $\sigma \in \mathbb{R}$ *such that* $0 \leq s-\sigma \leq 1$, *we have the following approximation error estimate*

$$\|\mathcal{J}_{d,\eta} f - f\|_{H^\sigma(\mathbb{R}^d)} \lesssim \eta^{s-\sigma}\|f\|_{H^s(\mathbb{R}^d)}\,. \tag{95}$$

[2] Since a directed set [6,28] is countable or uncountable, the one-parameter family of functions $\{f\}_{\varepsilon\in J}$, called sequences (respectively subsequences) by abuse of language, must be understood as generalized sequences (respectively subsequences) such as nets (respectively subnets) in the sense of Moore–Smith (see, e.g., Chapter 4, Sections 11 and 12 in [50]) or filters (respectively, finer filters) in the sense of Cartan (see, e.g., Chapter 1, Section 6 in [5]).

Proof. The property $\mathcal{J}_{d,\eta} f = \chi_\eta * f \in \mathscr{C}^\infty(\mathbb{R}^d)$ for any $\eta > 0$ and any $f \in \mathcal{D}'(\mathbb{R}^d)$ comes from the following convolution property $\mathcal{D}'(\mathbb{R}^d) * \mathcal{D}(\mathbb{R}^d) \subset \mathscr{C}^\infty(\mathbb{R}^d)$. For the proof of (95), observe

$$\begin{aligned} \|\mathcal{J}_{d,\eta} f - f\|^2_{H^\sigma(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} d\xi \, (1+|\xi|^2)^\sigma \, |\widehat{\chi}(\eta\xi) - 1|^2 \, |\hat{f}(\xi)|^2 \\ &\leq \eta^{2(s-\sigma)} \int_{\mathbb{R}^d} d\xi \, \frac{|\widehat{\chi}(\eta\xi) - 1|^2}{(\eta\,|\xi|)^{2(s-\sigma)}} \, (1+|\xi|^2)^s \, |\hat{f}(\xi)|^2 \, . \end{aligned}$$

Using the Lebesgue dominated convergence theorem, this last estimate leads to (95), since $\widehat{\chi}$ is smooth at the origin with $\widehat{\chi}(0) = 1$. □

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