Hydro- & Magnetohydro-dynamic Turbulence

Cargèse Summer School

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Introduction

Hydrodynamic turbulence (incompressible neutral flow) as a first step to understand astro- and geo-physical systems (atmosphere, ocean surface, rivers, galaxies, protostellar disks wherever electromagnetic forces are subdominant...)

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- large & apparently random fluctuations of velocity and pressure
- strong mixing of the fluid
- instability characteristic; a weak initial noise can be significantly amplified (chaotic system with large number of degrees of freedom)

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- values of kinematic viscosity for some fluids $[cm^2/s]$

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- estimation of some Reynolds numbers :
  atmosphere; $\nu \sim 0.015 cm^2/s$, $U \sim 10 m/s$, $L \sim 15 m \rightarrow R_e \sim 10^7$
  water pipe; $\nu \sim 0.01 cm^2/s$, $U \sim 0.1 m/s$, $D \sim 5 cm \rightarrow R_e \sim 5000$

Statistical description

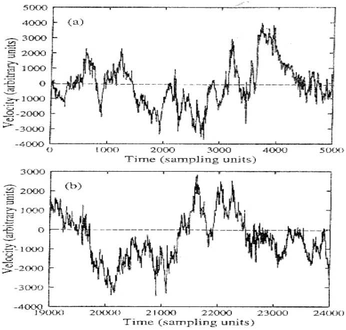

Fig. 3.1. One second of a signal recorded by a hot-wire (sampled at 5 kHz) in the S1 wind tunnel of ONERA (a); same signal, about four seconds later (b). Courtesy Y. Gagne and E. Hopfinger.

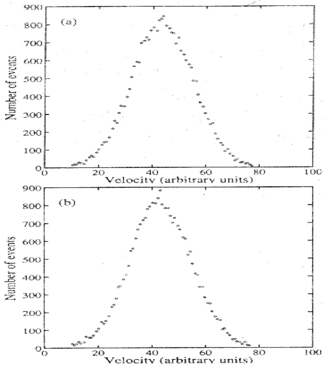

Fig. 3.3. Histogram for same signal as in Fig. 3.1(a), sampled 5000 times ov a time-span of 150 seconds (a); same histogram, a few minutes later (b).

2 signals recorded each 4"; quite similar but unpredictable in their detailed behaviors from (a) to (b)

pdf of recorded signals (few minutes later); essentially identical

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- signals may be decomposed into mean (i.e. time average) and fluctuating components

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{U}(\mathbf{x}) + \mathbf{u}'(\mathbf{x}, t) \text{ with } \mathbf{U}(\mathbf{x}) = \langle \mathbf{u}(\mathbf{x}, t) \rangle$$

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  determine *all* statistical properties of the fow is still a distant target ...
- preferable (theoretical point of view) to define "ensemble average" as an average over a very large number $\mathcal{N}$ of identical experiments of a given flow ($\mathcal{N}$ realisations): $\mathbf{U}(\mathbf{x}, t) = \langle \mathbf{u}(\mathbf{x}, t) \rangle_{\mathcal{N}}$

- result of ergodic theory: if $\mathbf{u}(\mathbf{x}, t)$ is a stationary random function of time (all ensemble quantities are time-independent) then (under certain mild subsidary conditions) ensemble average and time average yield the same result. Idem, if $\mathbf{u}(\mathbf{x}, t)$ is a stationary random function of space, ensemble average and space average give the same result

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- Homogeneous turbulence (1): statistical stationarity with respect to one or more coordinates, i.e. all satistical properties are invariant under translations of one or more coordinates.
  In practice, for ex., $\langle u_i(\mathbf{x}, t) u_j(\mathbf{x} + \mathbf{r}, t) \rangle = R_{ij}(\mathbf{r}, t)$, correlation tensor, is independent of $\mathbf{x}$ and only depends on $\mathbf{r}$

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- Isotropic turbulence (2): statistical stationarity with respect to all directions, i.e. all satistical properties are invariant under rotations.
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- Reflexionally symmetric turbulence (3): statistical stationarity under change from right-handed to left-handed frame of reference, i.e. all satistical properties are invariant under parity transformations.
  For ex., $R_{ij}(r, t)$ has no antisymmetric part ((1)+(2)+(3) or "full isotropy")

Spectral description of homogeneous turbulence

Suppose that $\mathbf{u}(\mathbf{x}, t)$ is a field of homogeneous turbulence with $\langle \mathbf{u} \rangle = 0$ (mean velocity suppressed by Galilean transformation) and consider its instantaneous structure (i.e. omit explicit time dependence from now on):

- Fourier transform is formally (as a generalised function) defined by $\hat{\mathbf{u}}(\mathbf{k}) = 1/(2\pi)^3 \iiint \mathbf{u}(\mathbf{x}) \mathrm{e}^{-i\mathbf{k}\cdot\mathbf{x}} d\mathbf{x} = \hat{\mathbf{u}}^*(-\mathbf{k})$ where $\hat{\mathbf{u}}(\mathbf{k})$ is Fourier amplitude, $\mathbf{k} = (k_x, k_y, k_z)$ wave-vector and $\mathbf{x} = (x, y, z)$ space-point

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- Second order correlation tensor and spectrum tensor of the velocity (Cramer's theorem for a stationary random process)

$$R_{ij}(\boldsymbol{x}) = \langle u_i(\boldsymbol{x}) u_j(\tilde{\boldsymbol{x}}) \rangle = \int \Phi_{ij}(\boldsymbol{k}) \mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{r}} d\boldsymbol{k} \quad \text{with } \tilde{\boldsymbol{x}} = \boldsymbol{x} + \boldsymbol{r}$$

$$\Phi_{ij}(\boldsymbol{k}) = 1/(2\pi)^3 \int R_{ij}(\boldsymbol{r}) \mathrm{e}^{-i\boldsymbol{k}\cdot\boldsymbol{r}} d\boldsymbol{r} \qquad \text{where } \int \cdot \, d\boldsymbol{x} \equiv \iiint \cdot \, d\boldsymbol{x}$$

$\Phi_{ij}(\boldsymbol{k})$ is a complex tensor such as $\int |\Phi_{ij}(\boldsymbol{k})| d\boldsymbol{k} < \infty$ and $\Phi = X_i X_j^* \Phi_{ij}(\boldsymbol{k})$ quadratic form $\geqslant 0$ ($\forall$ $\mathbf{X} \in \mathbb{C}$)

- Homogeneity $\longrightarrow$ $R_{ij}(\mathbf{r}) = R_{ji}(-\mathbf{r})$ and $\Phi_{ij}^*(\mathbf{k}) = \Phi_{ij}(-\mathbf{k}) = \Phi_{ji}(\mathbf{k})$

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- Remark :

$$\langle \hat{u}_i^*(\mathbf{k}) \hat{u}_j(\tilde{\mathbf{k}}) \rangle = 1/(2\pi)^6 \langle \iint u_i(\mathbf{x}) u_j(\tilde{\mathbf{x}}) \mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}} \mathrm{e}^{-i\tilde{\mathbf{k}}\cdot\tilde{\mathbf{x}}} d\mathbf{x} d\tilde{\mathbf{x}} \rangle$$

$$= 1/(2\pi)^6 \iint \langle u_i(\mathbf{x}) u_j(\tilde{\mathbf{x}}) \rangle \mathrm{e}^{i\mathbf{k}\cdot\mathbf{x} - i\tilde{\mathbf{k}}\cdot\tilde{\mathbf{x}}} d\mathbf{x} d\tilde{\mathbf{x}}$$

$$= 1/(2\pi)^6 \iint R_{ij}(\mathbf{r}) \mathrm{e}^{-i\tilde{\mathbf{k}}\cdot\mathbf{r}} \mathrm{e}^{i(\mathbf{k}-\tilde{\mathbf{k}})\cdot\mathbf{x}} d\mathbf{x} d\mathbf{r}$$

$$= 1/(2\pi)^3 \int R_{ij}(\mathbf{r}) \mathrm{e}^{-i\tilde{\mathbf{k}}\cdot\mathbf{r}} \quad 1/(2\pi)^3 \int \mathrm{e}^{i(\mathbf{k}-\tilde{\mathbf{k}})\cdot\mathbf{x}} d\mathbf{x} d\mathbf{r} = \Phi_{ij}(\tilde{\mathbf{k}}) \delta(\mathbf{k} - \tilde{\mathbf{k}})$$

Energy spectrum function

$$E = 1/2\langle \mathbf{u}(\mathbf{x})^2 \rangle = 1/2 \iint \langle \hat{u}_i^*(\mathbf{k})\hat{u}_i(\tilde{\mathbf{k}})\rangle \mathrm{e}^{i\mathbf{k}\cdot\mathbf{x} - i\tilde{\mathbf{k}}\cdot\tilde{\mathbf{x}}} d\mathbf{k} d\tilde{\mathbf{k}}$$

$$= 1/2 \iint \Phi_{ii}(\tilde{\mathbf{k}})\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x} - i\tilde{\mathbf{k}}\cdot\tilde{\mathbf{x}}} \delta(\mathbf{k} - \tilde{\mathbf{k}}) d\mathbf{k} d\tilde{\mathbf{k}} = 1/2 \int \Phi_{ii}(\mathbf{k}) d\mathbf{k}$$

Spectral density of energy is $\mathcal{E}(\mathbf{k}) \equiv 1/2\Phi_{ii}(\mathbf{k})$. If isotropy is assumed, no dependence on direction of **r** or **k**, angle averaging on sphere $S(k)$ of radius $k = |\boldsymbol{k}|$ in $\boldsymbol{k}$-space gives

$$E(k) = 4\pi k^2 \mathcal{E}(\mathbf{k}) = 1/2 \int_{S(k)} \Phi_{ii}(\mathbf{k}) dS \longrightarrow \int_0^\infty E(k)dk = 1/2\langle \mathbf{u}(\mathbf{x})^2 \rangle$$

$E(k)dk$ may be interpreted as the contribution to turbulent energy from a spherical annulus $(k, k + dk)$ of wave-numbers $k = |\boldsymbol{k}|$

- Enstrophy spectrum function
  with vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{u} = \int i\mathbf{k} \times \hat{\mathbf{u}}(\mathbf{k})\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}} d\mathbf{x}$

$$\Omega = 1/2\langle \boldsymbol{\omega}^2(\mathbf{x})\rangle = 1/2\int \Omega_{ii}(\boldsymbol{k})d\boldsymbol{k} = 1/2\int k^2\Phi_{ii}(\boldsymbol{k})d\boldsymbol{k}$$

  For isotropic turbulence $1/2\langle \boldsymbol{\omega}^2(\mathbf{x})\rangle = \int_0^\infty \Omega(k)dk = \int_0^\infty k^2 E(k)dk$

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- Helicity spectrum function

$$H^c = \langle \boldsymbol{u}(\boldsymbol{x}) \cdot \boldsymbol{\omega}(\boldsymbol{x})\rangle = i\epsilon_{ijm} \int k_j \Phi_{im}(\boldsymbol{k}) d\boldsymbol{k} = \int \mathcal{H}(\boldsymbol{k}) d\boldsymbol{k}$$

  Note that helicity is a pseudo-scalar. For isotropic turbulence
  $H(k) = 4\pi k^2 \mathcal{H}(\mathbf{k}) \longrightarrow \quad \langle \boldsymbol{u} \cdot \boldsymbol{\omega}\rangle = \int_0^\infty H(k)dk$

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  $H(k) = 4\pi k^2\mathcal{H}(\mathbf{k}) \longrightarrow \quad \langle\boldsymbol{u}\cdot\boldsymbol{\omega}\rangle = \int_0^\infty H(k)dk$
- Form of the spectrum tensor $\Phi_{ij}(\boldsymbol{k})$ for isotropic turbulence

$$\Phi_{ij}(\boldsymbol{k}) = \frac{E(k)}{4\pi k^2}\left[\delta_{ij} - \frac{k_ik_j}{k^2}\right] + i\frac{H(k)}{8\pi k^4}\epsilon_{ijm}k_m$$

  If, moreover, turbulence is reflexionally symmetric (or fully isotropic),
  $H(k) = 0$ and $\Phi_{ij}(\boldsymbol{k}) = \frac{E(k)}{4\pi k^2}\left[\delta_{ij} - \frac{k_ik_j}{k^2}\right]$

Phenomenological description (K41)

Consider an idealised situation: a statistically stationary flow forced by a volume force $\mathbf{F}$ which is stationary, random,homogeneous and fully isotropic, on characteristic scale $\ell_0 \sim 1/k_0$. The velocity obeys the usual Navier-Stokes equation (with density $\rho = cst$, and ν kinematic viscosity)

$$\partial \boldsymbol{u}/\partial t + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\frac{1}{\rho}\nabla p + \mathbf{F} + \nu \Delta \boldsymbol{u}$$

$$\nabla \cdot \boldsymbol{u} = 0$$

$$B.C.$$

What happens to the injected energy ?

$$\langle \frac{1}{2}\partial_t \boldsymbol{u}^2 + \boldsymbol{u} \cdot (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} \rangle = \langle \frac{1}{\rho}\boldsymbol{u} \cdot \nabla p + \mathbf{F} \cdot \boldsymbol{u} + \nu \boldsymbol{u} \cdot \Delta \boldsymbol{u} \rangle$$

$$\partial_t \frac{1}{2}\langle \boldsymbol{u}^2 \rangle = \langle \mathbf{F} \cdot \boldsymbol{u} \rangle - \nu \langle \boldsymbol{\omega}^2 \rangle$$

with $\langle \cdot \rangle$ ensemble, or time or space average, and using the following results implied by icompressibility and homogeneity

$$\langle \boldsymbol{u}\cdot(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\rangle = \langle(\boldsymbol{u}\cdot\nabla)\frac{1}{2}\boldsymbol{u}^2/\rangle = \nabla\cdot\langle\frac{1}{2}\boldsymbol{u}\boldsymbol{u}^2\rangle = 0$$

$$\langle \boldsymbol{u}\cdot\nabla p/\rho\rangle = \nabla\cdot\langle\boldsymbol{u}/\rho\rangle = 0$$

$$\langle \boldsymbol{u}\cdot\Delta\boldsymbol{u}\rangle = \langle\nabla\cdot(\boldsymbol{\omega}\times\boldsymbol{u}) - \boldsymbol{\omega}^2\rangle = -\langle\boldsymbol{\omega}^2\rangle$$

- For a turbulent flow statistically stationary with respect to time, i.e. $\frac{1}{2}\langle\boldsymbol{u}^2\rangle = cst$,

$$\underbrace{\langle\mathbf{F}\cdot\boldsymbol{u}\rangle}_{energy\ \ injection\ \ mean\ \ rate} \quad = \quad \underbrace{\nu\langle\boldsymbol{\omega}^2\rangle}_{energy\ \ dissipation\ \ mean\ \ rate} \quad = \epsilon$$

$$\langle \boldsymbol{u} \cdot (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} \rangle = \langle (\boldsymbol{u} \cdot \nabla)\frac{1}{2}\boldsymbol{u}^2 / \rangle = \nabla \cdot \langle \frac{1}{2}\boldsymbol{u}\boldsymbol{u}^2 \rangle = 0$$

$$\langle \boldsymbol{u} \cdot \nabla p/\rho \rangle = \nabla \cdot \langle \boldsymbol{u}/\rho \rangle = 0$$

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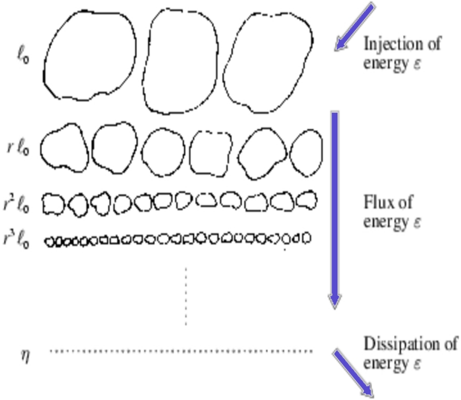
- Energy dissipation cannot arise at scale ℓ_0, at that scale, the Reynods number is very large and dissipation is very small $\Longrightarrow$ physical picture of the Richardson's energy cascade (1926)

Scenario of the energy cascade and viscous cut-off: Richardson's cascade



2 basic assumptions within the inertial range: scale-invariance (space-filling eddies, $0 < r < 1$) & localness of interactions (energy flux at scales $\sim \ell$ mainly involves scales of comparable size)

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Phenomenological tools to describe turbulent flow properties on scales ℓ within the inertial range ($\ell_\nu \ll \ell \ll \ell_0$) with typical velocity u_{ℓ},

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- energy flux on scale $\sim \ell$:
  $\epsilon_\ell \sim K_\ell / t_\ell \sim u_\ell^3 / \ell \sim \epsilon \sim u_0^3 / \ell_0 \rightarrow u_\ell \sim (\epsilon \ell)^{1/3}$
  and $\rightarrow u_\ell \sim u_0 (\ell / \ell_0)^{1/3}$

Characteristic scales of the flow

- dissipation scale: $t_{\ell} \sim t_\nu \;\rightarrow\; \ell_\nu/u_{\ell_\nu} \sim \ell_\nu^2 \;\rightarrow\; \ell_\nu \sim \nu/u_{\ell_\nu}$
  - with $u_{\ell_\nu} \sim \epsilon^{1/3}\ell_\nu^{1/3} \;\rightarrow \ell_\nu \sim (\nu^3/\epsilon)^{1/4}$
  - with $u_{\ell_\nu} \sim u_0(\ell_\nu/\ell_0)^{1/3} \;\rightarrow \ell_\nu \sim \ell_0 {R_e}^{-3/4} \rightarrow R_e \sim (\ell_0/\ell_\nu)^{4/3}$
- "Taylor scale": $\lambda \sim (E/\Omega)^{1/2}$ (isotropic case)
  with $E = \langle \boldsymbol{u}^2\rangle/2 \sim {u_0}^2/2$ & $\Omega = \langle \boldsymbol{\omega}^2\rangle/2 = \frac{1}{2}\epsilon/\nu \rightarrow \lambda \sim (E\nu/\epsilon)^{1/2}$
  or $\lambda \sim \ell_0 {R_e}^{-1/2}$ and $R_\lambda \sim u_0\lambda/\nu \sim R_e^{1/2}$
- integral scale: $\ell_0 \sim u_0^3/\epsilon \sim E^{3/2}/\epsilon$
- estimation of the number of degrees of freedom: $N \sim {\ell_0}^3/\ell_\nu^3 \sim R_e^{9/4}$, correct if motions at inertial range are fully disorganized, but coherent structures, vortex filaments,.. do exist thus the presence of some order

Energy and Enstrophy spectra for isotropic turbulence

- energy spectrum $E(k)$; $E = 1/2\langle \mathbf{u}(\mathbf{x})^2 \rangle = \int_0^\infty E(k)dk$
  the **k**-space is splitted into spherical shells, for ex. $k_0 2^{p-1} < k < k_0 2^p$, and one can write

  $$E = \sum_p K_p = \sum_p E(k_p)k_p \quad \text{(by dimensional consistency)}$$

  shell by shell, it gives

  $$K_p \sim u_p^2/2 \sim (\epsilon/k_p)^{2/3} \sim E(k_p)k_p \rightarrow E(k_p) \sim \epsilon^{2/3} k_p^{-5/3}$$

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- enstrophy spectrum $\Omega(k)$ in the inertial range

  $$\boxed{\Omega(k) = k^2 E(k) = C\epsilon^{2/3}k^{1/3}}$$

MHD approximation

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- MHD approximation for some plasmas and liquid metals

  * quasi-neutral property ($\nabla \cdot \mathbf{E} \simeq 0$, with **E** the electric field)

  * fluid approximation: electrical conduction of the medium by electrons alone

  * non relastivistic limit (typical velocity $U \ll c$)

  * collisional plasma/fluid: conductivity is independent of U (time evolution of the fluid $\gg$ time between 2 collisions ions/electrons, fluid elements contain many ions & electons)

  * trajectories of electrons are not changed under the magnetic field **B** action

MHD equations

- Maxwell's equations

  Faraday's law $\nabla \times \mathbf{E} + \partial \boldsymbol{B}/\partial t = 0$

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- note that $\nabla \times (\boldsymbol{u} \times \boldsymbol{B}) = -(\boldsymbol{u} \cdot \nabla)\boldsymbol{B} + (\boldsymbol{B} \cdot \nabla)\boldsymbol{u}$ represents both advection and stretching of the field $\boldsymbol{B}$ (with $\nabla \cdot \boldsymbol{u} = 0 = \nabla \cdot \boldsymbol{B}$)

- Incompressible MHD equations

$$\begin{aligned} \partial \boldsymbol{u}/\partial t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} &= -\nabla p/\rho + \nu\Delta\boldsymbol{u} + (\boldsymbol{j}\times\boldsymbol{B})/\rho + \mathbf{F} \\ \partial \boldsymbol{B}/\partial t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{B} &= (\boldsymbol{B}\cdot\nabla)\boldsymbol{u} + \eta\Delta\boldsymbol{B} \\ \nabla\cdot\boldsymbol{u} = 0 &= \nabla\cdot\boldsymbol{B} \quad \text{and} \quad B.C. \end{aligned}$$

where $\boldsymbol{j}\times\boldsymbol{B} = (\nabla\times\boldsymbol{B})\times\boldsymbol{B}/\mu_0$ is the Lorentz force and $\mathbf{F}$ a body force (friction, gravity, Coriolis force,..). $\boldsymbol{B}$ can be replaced by the scaled magnetic field $\boxed{\boldsymbol{b} = \boldsymbol{B}/(\rho\mu_0)^{1/2} = \boldsymbol{v_a}}$ which has dimension of a velocity and is called the Alfvén velocity although $\boldsymbol{B}$ is a pseudovector

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- dimensionless parameters
  * magnetic Reynolds number = induction / diffusion = $R_M = UL_0/\eta$
  * magnetic Prandtl numer $P_M = \nu/\eta = R_M/Re$, $P_M \ll 1$
  ($P_M \sim 10^{-7} - 10^{-2}$, protostellar disk, sun, and liquid sodium experiments $P_M \sim 10^{-6}$) or $P_M \gg 1$ ($P_M \sim 10^{14} - 10^{19}$ as in solar wind, protogalaxies, interstellar medium..)

- MHD equations in Elsässer variables

  The velocity $\boldsymbol{u}$ and magnetic field $\boldsymbol{b}$ can be combined into the Elsässer fields $\boldsymbol{z}^{\pm} = \boldsymbol{u} \pm \boldsymbol{b}$, to obtain more symmetric equations

$$(\partial_t + \boldsymbol{z}^{\mp} \cdot \nabla)\boldsymbol{z}^{\pm} = \nu_1 \Delta \boldsymbol{z}^{\pm} + \nu_2 \Delta \boldsymbol{z}^{\mp} - \nabla P_* + \boldsymbol{f}^{\pm}$$

  where $\nabla \cdot \boldsymbol{z}^{\pm} = 0$, $P_* = (p/\rho + \boldsymbol{b}^2/2)$ is the total pressure, and $\nu_1 = \frac{1}{2}(\nu + \eta)$, $\nu_2 = \frac{1}{2}(\nu - \eta)$

Ideal invariants in homogeneous MHD turbulence

- $\partial_t E^T = -\nu < \boldsymbol{\omega}^2 > - \eta < \boldsymbol{j}^2 >$, for $\nu = \eta = 0$ → total energy (kinetic + magnetic) is a conserved quantity

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- note that: 1) helicities are pseudo-scalars, 2) in an ideal fluid, the mutual topologies of tubes are conserved

Alfvén waves

- Linearization of incompressible MHD eqs around a uniform magnetic field $\boldsymbol{b}_0$ with $\rho_0 = cst, p_0 = cst, \boldsymbol{u}_0 = 0$ (ν and η neglected) leads to :

$$\partial_t \boldsymbol{z}^+ - (\boldsymbol{b}_0 \cdot \nabla)\boldsymbol{z}^+ = 0$$
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Looking for a solution of plane-wave type for perturbations

$$\boldsymbol{z}^\pm = \boldsymbol{z_k}^\pm \mathrm{e}^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \bar{\omega}^\pm t)}$$

gives: $\bar{\omega}^+ = -(\boldsymbol{b}_0 \cdot \boldsymbol{k})$ & $\bar{\omega}^- = +(\boldsymbol{b}_0 \cdot \boldsymbol{k})$ with $\boldsymbol{k} \cdot \boldsymbol{z}_k^+ = 0$ & $\boldsymbol{k} \cdot \boldsymbol{z}_k^- = 0$ (incompressibility).

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- $\boldsymbol{z}^+$ and $\boldsymbol{z}^-$ are the so-called Alfvén waves : transverse waves $(\boldsymbol{z}_k^\pm \perp \boldsymbol{k})$ with group velocity $v_g = \pm b_0$ and phase velocity $v_\phi = \pm b_0 k_\parallel / k$ (semi-dispersives waves), where $k_\parallel$ is the component of $\boldsymbol{k} \parallel \boldsymbol{b}_0$.

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- $\boldsymbol{z}^+$ and $\boldsymbol{z}^-$ are the so-called Alfvén waves : transverse waves ($\boldsymbol{z}_k^\pm \perp \boldsymbol{k}$) with group velocity $v_g = \pm b_0$ and phase velocity $v_\phi = \pm b_0 k_\parallel / k$ (semi-dispersives waves), where $k_\parallel$ is the component of $\boldsymbol{k} \parallel \boldsymbol{b}_0$.
- oppositely travelling waves: $\boldsymbol{z}^-$ travels in the $\boldsymbol{b}_0$-direction while $\boldsymbol{z}^+$ is backward travelling, with group velocity $\boldsymbol{b}_0$, the so-called Alfvén velocity denoted $\boldsymbol{v_a}$

* A uniform magnetic field $\boldsymbol{b}_0$ (or a local one at scale larger than a given ℓ in the inertial range, or at large scale) has a significant dynamical effect for energy transfers : $\boldsymbol{z}^+$ and $\boldsymbol{z}^-$ blob disturbances (wavepackets) only interact when they collide $\longrightarrow$ weakening of the transfer of energy between scales (i.e. weak nonlinearity)

* Multiple collisions are needed to pass energy in the blobs to smaller scales

* This is the basic idea of "IK" phenomenology (Iroshnikov 63, Kraichnan 65): interplay between turbulent eddies and Alfvén waves travelling along a mean field $\longrightarrow$ crucial difference between hydrodynamic and conducting fluids

* Does Kolmogorov's approach still work ? Does it need to be modified ? Alfvén waves and correlation between $\boldsymbol{u}$ and $\boldsymbol{b}$ fields (cross helicity) are crucial and lead to a lack of universality for inertial MHD spectra

Phenomenologies

Let's take $P_M \sim 1$ from now on.
Suppose $|\boldsymbol{b}| \ll |\boldsymbol{b}_0|$, the IK phenomenology is based on weak nonlinear interactions and many collisions, say N, between z^+ and z^- wavepackets of similar size ℓ, are needed to pass energy to smaller scales. For simplicity, ignore anisotropy ($\ell_{||} \sim \ell_\perp \sim \ell$) and suppose zero cross helicity $H^C \sim 0$ ($z_\ell^+ \sim z_\ell^- \sim z_\ell$). Disturbances are sheared by an amount

$$\delta z_\ell \sim (z_\ell z_\ell/\ell)(\ell/b_0) \longrightarrow \delta z_\ell/z_\ell \sim z_\ell/b_0$$

- $t_a \sim \ell/b_0 \equiv \ell/v_a$ is the interaction time for one collision (Alfvén time) at scale ℓ, i.e. characteristic time for propagation over a distance ℓ

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- where t_ℓ is the advection time at scale ℓ; $t_\ell \sim \ell/z_\ell$

Isotropic descriptions

- uncorrelated case $<\boldsymbol{u}\cdot\boldsymbol{b}>\sim<(z^+)^2-(z^-)^2>\sim 0$, $z^+_\ell \sim z^-_\ell \sim z_\ell$

  * K41, $b_0 \sim 0$

  $t^+_{tr} \sim t^-_{tr} \sim t_\ell \sim \ell/z_\ell \longrightarrow \epsilon_\ell \sim \epsilon \sim {z_\ell}^3/\ell$ within inertial range

  $K^\pm_\ell \sim {z_\ell}^2 \sim kE(k)$, $\epsilon_\ell \sim [kE(k)]^{3/2}k \sim \epsilon$ $\boxed{E(k) \sim \epsilon^{2/3}k^{-5/3}}$

  * dissipation scale $t_{tr} \sim t_\nu \sim \ell^2/\nu \rightarrow \ell_\nu \sim (\nu^3/\epsilon)^{1/4}$ $(P_M \sim 1)$

  * IK $b_0 \gg u_\ell \sim b_\ell$, $t_a \ll t_\ell$

  $t^+_{tr} \sim t^-_{tr} \sim t_{tr} \sim {t_\ell}^2/t_a \sim \ell b_0/z^2_\ell \rightarrow \epsilon^+_\ell \sim \epsilon^-_\ell \sim \epsilon_\ell \sim \epsilon \sim z^4_\ell/\ell b_0$

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- correlated case $< \boldsymbol{u} \cdot \boldsymbol{b} >\sim< (z^+)^2 - (z^-)^2 >\nsim 0$, $z_\ell^+ \nsim z_\ell^-$

  * IK $t_a \ll t_\ell^\pm$ $t_\ell^+ \sim \ell/z_\ell^-$, $t_\ell^- \sim \ell/z_\ell^+$

  $t_{tr}^+ \sim {t_\ell^+}^2/t_a \sim \ell b_0/{z_\ell^-}^2 \to \epsilon_\ell^+ \sim {z_\ell^+}^2/t_{tr}^+ \sim {z_\ell^+}^2{z_\ell^-}^2/\ell b_0$

  $t_{tr}^- \sim {t_\ell^-}^2/t_a \sim \ell b_0/{z_\ell^+}^2 \to \epsilon_\ell^- \sim {z_\ell^-}^2/t_{tr}^- \sim {z_\ell^-}^2{z_\ell^+}^2/\ell b_0$

$K_\ell^+ \sim {z_\ell^+}^2 \sim kE^+(k) \rightarrow z_\ell^+ \sim \sqrt{kE^+(k)}$,
$K_\ell^- \sim {z_\ell^-}^2 \sim kE^-(k) \rightarrow z_\ell^- \sim \sqrt{kE^-(k)}$
within the inertial range $k_0 \ll k \ll k_\nu$
$\epsilon_\ell^+ \sim \epsilon_\ell^- \sim \epsilon \sim [kE^+(k)][kE^-(k)]k/b_0$ $\boxed{E^+(k)E^-(k) \sim (b_0\epsilon)k^{-3}}$
suppose that $E^+(k) \sim k^{-m^+}$ and $E^-(k) \sim k^{-m^-}$ $\longrightarrow$ $\boxed{m^+ + m^- = 3}$
* dissipation scales ($P_M \sim 1$)
$t_{tr}^+ \sim t_\nu^+ \sim {\ell^+}^2/\nu \rightarrow \ell_\nu^+ \sim \nu b_0/{z_{\ell_\nu^+}^-}^2$, $t_{tr}^- \sim t_\nu^- \sim {\ell^-}^2/\nu \rightarrow \ell_\nu^- \sim \nu b_0/{z_{\ell_\nu^-}^+}^2$
it can be showned that $k_\nu^+ \sim k_\nu^-$ which leads to $k_\nu \sim (\epsilon/b_0\nu^2)^{1/3} \sim 1/\ell_\nu$

* K41 $b_0 \sim 0$, a similar analysis
$t_{tr}^+ \sim t_\ell^+$, $t_{tr}^- \sim t_\ell^-$ and $\epsilon_\ell^+ \sim \epsilon_\ell^- \sim \epsilon$ leads to $m^+ = m^- = 5/3$
* with dissipation wave number $k_\nu \sim (\epsilon/\nu^3)^{1/4}$

Anisotropic descriptions

Here, let's write $\boldsymbol{B} = \boldsymbol{B}_0 + \boldsymbol{b}$, where $\boldsymbol{B}_0$ is an ambient magnetic field.

It is possible to take into account anisotropy within weak turbulence theory (weak nonlinearity) using resonant triad waves interactions [1] theory; waves satisfy conditions:

$\boldsymbol{k}^{(1)} + \boldsymbol{k}^{(2)} = \boldsymbol{k}^{(3)}$, $\bar{\omega}^{(1)} + \bar{\omega}^{(2)} \approx \bar{\omega}^{(3)}$, with dispersion relationship $\bar{\omega} = \pm v_a k_{\|}$

As only oppositely travelling waves interact, the 3 waves must satisfy

$k_{\|}^{(1)} + k_{\|}^{(2)} = k_{\|}^{(3)}$ and $v_a k_{\|}^{(1)} - v_a k_{\|}^{(2)} \approx \pm v_a k_{\|}^{(3)}$,

the only possibilities are

$$k_{\|}^{(1)} \approx k_{\|}^{(3)}, \quad k_{\|}^{(2)} \approx 0, \quad \bar{\omega}^{(2)} \approx 0$$

$$k_{\|}^{(2)} \approx k_{\|}^{(3)}, \quad k_{\|}^{(1)} \approx 0, \quad \bar{\omega}^{(1)} \approx 0$$

- modes $k_{\|} \approx 0$, $\bar{\omega} \approx 0$ are not really waves but rather quasi-2D fluctuations highly elongated along $\boldsymbol{B}_0$
- wave (1), for ex., interacts with a quasi-static quasi-2D disturbance and the generated wave (3) has $\sim k_{\|}^{(1)}$ so negligible change in $\ell_{\|}$ from the collision

---

[1] strict resonance is not required for the non-linear interactions between 3 waves of form $\boldsymbol{z}_k e^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \bar{\omega} t)}$

- For sake of simplicity, we still suppose $P_M \sim 1$ and zero cross helicity ($H^c \sim 0$), thus $z_{\boldsymbol{\ell}}^{+} \sim z_{\boldsymbol{\ell}}^{-} \sim z_{\boldsymbol{\ell}}$

- For sake of simplicity, we still suppose $P_M \sim 1$ and zero cross helicity ($H^c \sim 0$), thus $z_{\ell}^{+} \sim z_{\ell}^{-} \sim z_{\ell}$
- IKa, $B_0 \gg b_{rms}$

  * $t_a \ll t_{\ell}$

  energy transfer time

  $t_{tr} \sim t_{\ell}^2/t_a \sim (\ell_{\perp}/z_{\ell})^2/(\ell_{\|}/B_0) \sim (k_{\|}B_0)/(k_{\perp}^2 z_{\ell}^2)$

  energy flux down through the inertial range

  $\epsilon_{\ell}^{+} \sim \epsilon_{\ell}^{-} \sim \epsilon_{\ell} \sim \epsilon \sim z_{\ell}^2/t_{tr} \sim k_{\perp}^2 z_{\ell}^4/k_{\|}B_0 \longrightarrow z_{\ell} \sim (\epsilon k_{\|}B_0/k_{\perp}^2)^{1/4}$

  which leads to $\epsilon \sim k_{\perp}^2 (k_{\|}k_{\perp}E(k_{\perp},k_{\|}))^2/(k_{\|}B_0)$ and

  $\boxed{E(k_{\perp},k_{\|}) \sim (\epsilon B_0)^{1/2} k_{\|}^{-1/2} k_{\perp}^{-2}}$ (Ng & Bhattacharjee, 1997)

  * $t_a \ll \epsilon t_{\ell}$, asymptotic analytical result within Alfvén waves turbulence theory $\boxed{E(k\,\|, k\,\perp) \sim C_k f(k_{\|}) k_{\perp}{}^{-2}}$ $(k_{\|} \neq 0)$ (Galtier et al., 2000)

  (no energy transfer along $\boldsymbol{B}_0$)

- K41a, $B_0 \sim b_{rms}$

  "strong" turbulence regime, i.e. strong non-linear collisions of z^+ and z^- propagating waves to pass energy to smaller scales, with the so called *critical balance* assumption $t_\ell \sim t_a$, i.e. equilibrium between inertial forces and Maxwell stresses (Goldreich & Sridhar, 1995)
  * non-linear interaction time = interaction time of 2 oppositely travelling waves (as only 1 collision is needed): $t_a \sim \ell_{||}/B_0$
  * flux of energy though inertial rang: $\epsilon_\ell \sim z_\ell^2/t_a \sim z_\ell^2/t_\ell \sim z_\ell^3/\ell_\perp$
  * this yields $z_\ell^2 \sim \epsilon^{2/3}\ell_\perp^{2/3} \rightarrow z_\ell^2 \sim k_\perp E(k_\perp) \sim \epsilon^{2/3}{k_\perp}^{-2/3}$ and thus
  $\boxed{E(k_\perp) \sim \epsilon^{2/3}{k_\perp}^{-5/3}}$

  Remarks:
  \- $\ell_{||} \sim B_0\ell_\perp/z_\ell \sim (B_0/\epsilon^{1/3})\ell_\perp^{2/3}$
  \- $z_\ell^2 \sim \epsilon^{2/3}\ell_\perp^{2/3} \sim \epsilon\ell_{||}/B_0 \longrightarrow E(k_{||}) \sim (\epsilon/B_0)k_{||}^{-2}$
  \- within IK theory ($t_a \ll t_\ell$), assuming $E(k_\perp, k_{||}) \sim k_\perp^{-a}k_{||}^{-b}$, it can be show that $3a + 2b = 7$, thus $a = 5/3$, $b = 1$ for K41a & $a = 2$, $b = 1/2$ for IKa, and, if $t_a(\ell_{||})/t_\ell(\ell_\perp) \sim cst$, $\ell_{||} \sim (B_0/\epsilon_{IKa}^{1/3})\ell_\perp^{2/3}$ (Galtier et al., 2005)

von Kármán-Howarth equations

To obtain such von Kármán-Howarth (VKH) equations:

1) write the two-point (at $\boldsymbol{x}$ & $\boldsymbol{x}+\boldsymbol{r}$) correlations for the different components of given fields ($\boldsymbol{u}$,$\boldsymbol{b}$, $z^{\pm}$, ..), or their respective increments, namely 1st, 2nd and 3rd order correlations, reduce the associated tensors (or pseudo-tensors) using incompressibility condition, homogeneity and isotropy assumptions, finally write the tensor coefficients in terms of u_p longitudinal component ($\|$ to $\boldsymbol{r}$) and u_{n1}, u_{n2} lateral components ($\perp$ to $\boldsymbol{r}$)

2) write the movement equations at two different spatial locations, $\boldsymbol{x}$ & $\boldsymbol{x}+\boldsymbol{r}$, derive the time evolution of the two-point second order correlation of the fields ($\boldsymbol{u}$,$\boldsymbol{b}$, $z^{\pm}$, ..) and, using homogeneity, obtain the equations for the tensor coefficients

- VKH eq. (1938) for homogeneous fully isotropic NS turbulence

$$\frac{\partial}{\partial t} < u_p(\boldsymbol{x}) u_p(\boldsymbol{x}+\boldsymbol{r}) >= \frac{1}{r^4}\frac{\partial}{\partial r}[r^4 < u_p^2(\boldsymbol{x}) u_p(\boldsymbol{x}+\boldsymbol{r}) >] + 2\nu \frac{1}{r^4}\frac{\partial}{\partial r}[r^4 \frac{\partial}{\partial r} < u_p(\boldsymbol{x}) u_p(\boldsymbol{x}+\boldsymbol{r}) >] \quad (1)$$

a VKH eq. for helical flows (skew isotropy) can be derived (Gomez et al., 2000)

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- VKH for homogeneous isotropic MHD turbulence

for sake of simplicity, let consider $P_M = 1$ (Politano & Pouquet, 1998)

$$\frac{\partial}{\partial t} < z_p^{\pm}(\boldsymbol{x}) z_p^{\pm}(\boldsymbol{x}+\boldsymbol{r}) >= \frac{1}{r^4}\frac{\partial}{\partial r}[r^4 < z_p^{\pm}(\boldsymbol{x}) z_p^{\mp}(\boldsymbol{x}) z_p^{\pm}(\boldsymbol{x}+\boldsymbol{r}) >] + 2\nu \frac{1}{r^4}\frac{\partial}{\partial r}[r^4 \frac{\partial}{\partial r} < z_p^{\pm}(\boldsymbol{x}) z_p^{\pm}(\boldsymbol{x}+\boldsymbol{r}) >] \quad (2)$$

a VKH eq. for magnetic helicity can be obtained (Politano et al. 2003)

Laws for third-order correlation of increments

- Kolmogorov "4/5" law
  * consider velocity increments $\delta u_i(\boldsymbol{r}) = u_i(\boldsymbol{x}+\boldsymbol{r}) - u_i(\boldsymbol{x})$ and the 2nd order $< \delta u_i(\boldsymbol{r})\delta u_j(\boldsymbol{r}) >$ and 3rd order $< \delta u_i(\boldsymbol{r})\delta u_j(\boldsymbol{r})\delta u_k(\boldsymbol{r}) >$ structure functions. Let replace them in VKH eq. (1), and use $\partial_t E = -\epsilon = \frac{1}{2}\partial_t < u_i(\boldsymbol{x})u_i(\boldsymbol{x}) >= \frac{3}{2}\partial_t < u_p^2(\boldsymbol{x}) >$ (by isotropy)
  * under hypothesis : i) $t \to \infty$ (stationary state) and ϵ is finite per unit mass (ν fixed) and ii) $\nu \to 0$ (ϵ still fixed), one obtains

$$< (\delta u_p(\boldsymbol{r}))^3 >= -\frac{4}{5}\epsilon r \qquad \text{within the inertial range}$$

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- MHD "4/3" law

  a similar approach for MHD VKH eq. (2) gives in the inertial range

  $$< (\delta \boldsymbol{z}^{\pm} \cdot \delta \boldsymbol{z}^{\pm})\delta z_p^{\mp}(\boldsymbol{r}) >= -\frac{4}{3}\epsilon^{\pm} r$$

  (Politano & Pouquet, 1998)

Structures fonctions and scaling exponents

Two-points statistics can be described in terms of moments of velocity increments (or "structure functions") of order p, for $\ell \ll \ell_0$, namely

$$\delta v_{\boldsymbol{\ell}}^p \equiv < [u(\boldsymbol{x}+\boldsymbol{\ell}) - u(\boldsymbol{x})]^p >$$

where, here, u is the field component, say velocity, in the direction of the separation vector $\boldsymbol{\ell} = (\ell, 0, 0)$ (longitudinal component).

Suppose a scaling law within the inertial range $\ell_\nu \ll \ell \ll \ell_0$; $\boxed{\delta v_{\boldsymbol{\ell}}^p \sim \ell^{\xi_p}}$

exact results

* if fluctuations are bounded then $\xi_{2p+2} \geqslant \xi_{2p}$ (p =1, 2, 3, ...) (Frisch 91)

* Schwartz inequality gives $\xi_{p+q} \geqslant (\xi_{2p} + \xi_{2q})/2$ (for all positive p, q)

* hence, $d^2\xi_p/dp^2 \leqslant 0$ and ξ_p is a concave function of p ($\forall p > 0$)

linear behavior of ξ_p predicted from phenomenology

* K41 approach $\xi_p = p/3$

* IK approach $\xi_p = p/4$ (uncorrelated case)

Experimental results

* many analysis of observational and numerical data show departure from a linear behavior of the scaling exponents, ξ_p, and this departure becomes larger as $p \nearrow$ ... something is going wrong with the original K41 theory

* p.d.f.s of velocity increments have less and less Gaussian forms as $\ell \searrow$; for $\ell \sim \ell_0$ the p.d.f of this increments is essentially indistinguishable from a Gaussian, at inertial range separations, it develops almost expontial wings, and at even smaller scales, it takes form of "stretched exponential". This is probably due to the strong localization of the strong fluctuations

Interpretation and modeling

- Source of the fundamental problem with K41 : within a volume $\mathcal{V}_{\boldsymbol{\ell}}$, it is not the mean rate of dissipation ϵ that is relevant but rather the local dissipation $\epsilon(\boldsymbol{x},t) = \nu/2(\partial_j u_i + \partial_i u_j)^2$ averaged over $\mathcal{V}_{\boldsymbol{\ell}}$ (for ex. a sphere of center $\boldsymbol{x}$ and radius ℓ) : $\epsilon_{\boldsymbol{\ell}}(\boldsymbol{x},t) \equiv < \epsilon(\boldsymbol{x},t) >_{\mathcal{V}_{\boldsymbol{\ell}}}$
  $< \epsilon_{\boldsymbol{\ell}}^p(\boldsymbol{x},t) >$ will depend upon $\boldsymbol{\ell}$ (homogeneity) and let's suppose
  $\boxed{< \epsilon_{\boldsymbol{\ell}}^p(\boldsymbol{x},t) > \sim \boldsymbol{\ell}^{\tau_p}}$ $(\boldsymbol{\ell}_\nu \ll \boldsymbol{\ell} \ll \boldsymbol{\ell}_0)$

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- $(\delta v_{\boldsymbol{\ell}})^p \sim (\boldsymbol{\ell}\epsilon_{\boldsymbol{\ell}})^{p/3} \sim \boldsymbol{\ell}^{p/3} \boldsymbol{\ell}^{\tau_{p/3}} \sim \boldsymbol{\ell}^{\xi_p} \longrightarrow \quad \xi_p = p/3 + \tau_{p/3}$

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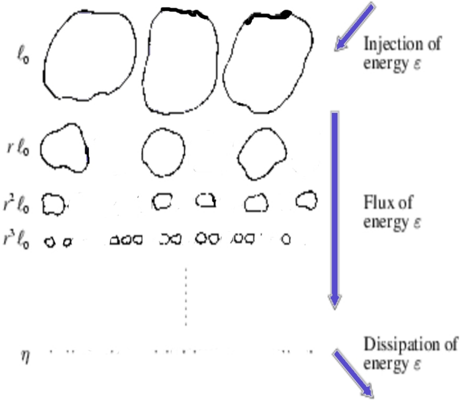
$< \epsilon_{\boldsymbol{\ell}}^p(\boldsymbol{x},t) >$ will depend upon ℓ (homogeneity) and let's suppose
$\boxed{< \epsilon_{\boldsymbol{\ell}}^p(\boldsymbol{x},t) > \sim \boldsymbol{\ell}^{\tau_p}}$ $(\boldsymbol{\ell}_\nu \ll \boldsymbol{\ell} \ll \boldsymbol{\ell}_0)$
- $\epsilon_{\boldsymbol{\ell}} \sim (\delta v_{\boldsymbol{\ell}})^2 / t_{\boldsymbol{\ell}} \sim (\delta v_{\boldsymbol{\ell}})^3 / \boldsymbol{\ell}$, with $\epsilon_{\boldsymbol{\ell}} \sim < \epsilon_{\boldsymbol{\ell}}(\boldsymbol{x},t) >$, $\delta v_{\boldsymbol{\ell}}$ has the same scaling laws than $(\boldsymbol{\ell}\epsilon_{\boldsymbol{\ell}})^{1/3}$ (Refined similarity hypothesis, Kolmogorov 62)
- $(\delta v_{\boldsymbol{\ell}})^p \sim (\boldsymbol{\ell}\epsilon_{\boldsymbol{\ell}})^{p/3} \sim \boldsymbol{\ell}^{p/3}\boldsymbol{\ell}^{\tau_{p/3}} \sim \boldsymbol{\ell}^{\xi_p} \longrightarrow \quad \xi_p = p/3 + \tau_{p/3}$
- many attempts to take into account the influence of possibly strong fluctuations in ϵ (or intermittency) with a modeling of $\tau_{p/3}$ exponent retaining the central concept of energy cascade through an extended inertial range (Log-normal model (Kolmogorov-Obukhov 62), $\beta-$model (Frisch et al. 78), Log-Poisson model (She-Lévêque 94))

Scenario of modified Richardson's cascade



sporadic energy tranfer through inertial range: only a small fraction of eddies of size $\ell \ll \ell_0$ is involved in the energy transfer to smaller scales, the other ℓ-eddies stay at rest (excitation on scale ℓ is confined, eddies are thus no more space-filling)

Log-Poisson model

The Log-Poisson model (She-Lévêque (SL),1994) currently provides the best fit for the ξ_p-exponents computed from experimental or numerical data.

- essential assumption: existence of a hierarchy of successive moments of energy dissipation at a given scale ℓ with a power law exponent, β, of the hierarchy $(0 < \beta < 1)$
- scaling exponent, α, for the characteristic time to dissipate the maximum amount of energy in the most intermittent dissipative structures; $t_\ell \sim \ell^\alpha$ (one can set a value for α in accordance with some phenomenology)
- C_0 codimension of the dissipative structures; $C_0 = \alpha/(1-\beta)$, and as $C_0 \leqslant D$ (where D is the dimension of space) $\rightarrow \beta \leqslant 1 - \alpha/D$

The model is thus a two-parameter model (for a general formulation of the model see Politano & Pouquet, 1995).

- SL HD $$\boxed{\xi_p = \frac{p}{3} + \alpha(\frac{1-\beta^{p/3}}{1-\beta} - \frac{p}{3})}$$

  "standard" model: $\alpha = 2/3$ (K41) and $C_0 = 2$ codimension of tube-like dissipative structures $\rightarrow \beta = 2/3$ (original SL model, 1994)

- SL MHD IK, case $H^c \sim 0$ & $P_M \sim 1$, $$\boxed{\xi_p = \frac{p}{4} + \alpha(\frac{1-\beta^{p/4}}{1-\beta} - \frac{p}{4})}$$

  "standard" model: $\alpha = 1/2$ (IK) and $C_0 = 1$ codimension of sheet-like dissipative structures $\rightarrow \beta = 1/2$ (Grauer et al., 1994)

- SL MHD K41, case $H^c \sim 0$ & $P_M \sim 1$, $$\boxed{\xi_p = \frac{p}{3} + \alpha(\frac{1-\beta^{p/3}}{1-\beta} - \frac{p}{3})}$$

  "standard" model: $\alpha = 2/3$ (K41) and $C_0 = 1$ codimension of sheet-like dissipative structures $\rightarrow \beta = 1/3$ (Horbury & Balogh, 1997)

In the case of anisotropic MHD see, for ex., W.-C. Müller, in Lecture Notes in Physics, vol. 756, 2009

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